Online Appendix for:

A Quantity-Driven Theory of Term Premia and Exchange Rates

Robin Greenwood Samuel G. Hanson

Jeremy C. Stein Adi Sunderam

Harvard University and NBER

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A Baseline model

A.1 Perpetuities

Consider a perpetual default-free bond which pays a coupon of C each period. Let P_t^y denote the price of this perpetuity at time t. The gross return on the perpetual bond from time t to $t+1$ is

$$1 + R_{t+1}^y = \frac{P_{t+1}^y + C}{P_t^y}.$$

To generate a tractable linear model, we use a Campbell-Shiller (1988) log-linear approximation to the return on this perpetuity. Specifically, defining

$$\delta \equiv 1/\left(1+C\right) < 1,$$

the one-period log return on the long-term bond is

$$r_{t+1}^y \equiv \ln\left(1 + R_{t+1}^y\right) \approx \overbrace{\frac{1}{1-\delta}}^{D} y_t - \overbrace{\frac{\delta}{1-\delta}}^{D-1} y_{t+1} = y_t - \frac{\delta}{1-\delta}\left(y_{t+1} - y_t\right), \tag{1}$$

where y_t is the log yield-to-maturity on the long-term bond at time t and where

$$D = \frac{1}{1-\delta} = \frac{C+1}{C}$$

is the Macaulay duration when the bond is trading at par.[1]

To derive this approximation note that the Campbell-Shiller (1988) approximation of the 1-period log return is

$$r_{t+1}^y = \ln\left(P_{t+1}^y + C\right) - p_t^y \approx \theta + \delta p_{t+1}^y + \left(1-\delta\right)c - p_t^y, \tag{2}$$

where $c = \log\left(C\right)$ and $\delta = 1/\left(1 + \exp\left(c - \overline{p}^y\right)\right)$ and $\theta = -\log\left(\delta\right) - \left(1-\delta\right)\log\left(\delta^{-1} - 1\right)$ are parameters of the log-linearization. Iterating equation (2) forward, the log bond price is

$$p_t^y = \left(1-\delta\right)^{-1}\theta + c - \textstyle\sum_{i=0}^{\infty}\delta^i E_t\left[r_{t+i+1}^y\right].$$

Applying this approximation to the *yield-to-maturity*, defined as the *constant return* that equates bond price and the discounted value of promised cashflows, we obtain

$$p_t^y = \left(1-\delta\right)^{-1}\theta + c - \left(1-\delta\right)^{-1} y_t. \tag{3}$$

Equation (1) then follows by substituting the expression for p_t^y in equation (3) into the Campbell-Shiller return approximation in equation (2).

Assuming the steady-state price of the bonds is par ($\overline{p}^y = 0$), we have $\delta = 1/\left(1+C\right)$. Thus, bond duration is $D = -\partial p_{L,t}/\partial y_t = \left(1-\delta\right)^{-1} = \left(1+C\right)/C$. Since

$$-\partial p_t^y/\partial y_t = -\left(\partial P_t^y/\partial Y_t\right)\left(\left(1+Y_t\right)/P_t^y\right) = \left(Y_t^{\cdot} + 1\right)/Y_t$$

this corresponds to Macaulay duration when the bonds are trading at par ($Y_t = C$).

[1] This log-linear approximation for default-free coupon-bearing bonds appears in Campbell, Lo, and MacKinlay (1997) and Campbell (2018).

A.2 Equilibrium conjecture

There are three prices that we need to pin down in equilibrium: y_t, y_t^*, and q_t. We conjecture that prices are a linear function of a state vector $\mathbf{z}_t$:

$$\begin{aligned} y_t &= \alpha_0^y + \boldsymbol{\alpha}_1^{y\prime}\mathbf{z}_t, \\ y_t^* &= \alpha_0^{y*} + \boldsymbol{\alpha}_1^{y*\prime}\mathbf{z}_t, \\ q_t &= \alpha_0^q + \boldsymbol{\alpha}_1^{q\prime}\mathbf{z}_t. \end{aligned}$$

Given our assumptions, the 5×1 state vector $\mathbf{z}_t = \left[i_t - \bar{\imath}, \;\; i_t^* - \bar{\imath}, \;\; s_t^y - \bar{s}^y, \;\; s_t^{y*} - \bar{s}^y, \;\; s_t^q\right]'$ follows a VAR(1) process $\mathbf{z}_{t+1} = \mathbf{\Phi}\mathbf{z}_t + \boldsymbol{\varepsilon}_{t+1}$, with $Var_t\left[\boldsymbol{\varepsilon}_{t+1}\right] = \mathbf{\Sigma}$ and Φ= $\mathbf{diag}(\phi_i, \phi_i, \phi_{s^y}, \phi_{s^y}, \phi_{s^q})$. We stack these three prices as a vector $\mathbf{y}_t = \mathbf{a}+ \mathbf{A}\mathbf{z}_t$, where $\mathbf{y}_t = [y_t, y_t^*, q_t]'$, $\mathbf{a} = [\alpha_0^y, \alpha_0^{y*}, \alpha_0^q]'$, $\mathbf{A} = [\boldsymbol{\alpha}_1^y, \boldsymbol{\alpha}_1^{y*}, \boldsymbol{\alpha}_1^q]'$.

A.3 Equilibrium concept, multiplicity, and selection

A rational expectations equilibrium of our overlapping-generations model is a fixed point of a specific operator involving the "price-impact" coefficients, $\boldsymbol{\alpha} = vec\left(\mathbf{A}\right)$, which show how the supplies impact bond yields and FX prices. Specifically, consider the operator $\mathbf{f}\left(\boldsymbol{\alpha}_0\right)$ which gives the price-impact coefficients that will clear the market for long-term bonds and FX when agents conjecture that $\boldsymbol{\alpha} = \boldsymbol{\alpha}_0$. Thus, a rational expectations equilibrium of our model is a fixed point $\boldsymbol{\alpha}^* = \mathbf{f}\left(\boldsymbol{\alpha}^*\right)$.

In any rational expectations equilibrium of our baseline model, bond yields always and FX prices reflect the expected path of future short rates. As a result, risk premia do not depend on short rates. This implies that an equilibrium of our baseline model is a solution to a system of 9 nonlinear equations in 9 unknowns. Specifically, we need to determine how equilibrium yields and FX prices respond to shifts in the supply of long-term bonds and the FX carry trade: this generates 9 unknowns and 9 corresponding equations.[2]

When supply is stochastic, an equilibrium solution only exists if investors are sufficiently risk tolerant (i.e., for τ sufficiently large). When an equilibrium exists, there are multiple equilibrium solutions. Equilibrium non-existence and multiplicity of this sort arise in overlapping-generations, rational-expectations models such as ours where risk-averse investors with finite investment horizons trade an infinitely-lived asset that is subject to supply shocks.[3] Different equilibria correspond to different self-fulfilling beliefs that investors can hold about the price-impact of supply shocks and, hence, the risks associated with holding long-term bonds and the FX carry trade.

The intuition for equilibrium multiplicity can be understood most clearly if short-lived investors hold a single long-lived asset. If investors are sufficiently risk tolerant there are two equilibria in this special case: a low price impact (or low return volatility) equilibrium and a high price impact (or high return volatility) equilibrium. If investors believe that supply shocks will have a large impact on prices, they will perceive the asset as being highly risky. As a result, investors will only absorb a positive supply shock if they are compensated by a large decline in prices, making the initial belief self-fulfilling. However, if investors believe that prices will be less sensitive to supply shocks, they will perceive the asset as being less risky and will absorb a supply shock even if they are only compensated by a modest decline in prices.

While our model admits multiple equilibria, we always find a unique equilibrium that is stable

[2]Once we allow bond supply to depend on the level of yields and carry trade exposures to depend on the exchange rate, risk premia will depend short rates. In that case, we need to determine how equilibrium yields and FX prices respond to shifts in the supply of long-term bonds and the FX carry trade as well as short rates: this generates 15 unknowns and 15 corresponding equations.

[3]For previous treatments of these issues, see Spiegel (1998), Bacchetta and van Wincoop (2003), Watanabe (2008), Banerjee (2011), Greenwood and Vayanos (2014), and Albagli (2015).

in the sense that equilibrium is robust to a small perturbation in investors' beliefs regarding the equilibrium that will prevail in the future. Formally, letting $\boldsymbol{\alpha}^{(1)} = \boldsymbol{\alpha}^* + \boldsymbol{\xi}$ for some small $\boldsymbol{\xi}$ and defining $\boldsymbol{\alpha}^{(n)} = \mathbf{f}(\boldsymbol{\alpha}^{(n-1)})$, an equilibrium $\boldsymbol{\alpha}^*$ is stable if $\lim_{n\to\infty} \boldsymbol{\alpha}^{(n)} = \boldsymbol{\alpha}^*$ and is unstable if $\lim_{n\to\infty} \boldsymbol{\alpha}^{(n)} \neq \boldsymbol{\alpha}^*$. Let $\{\lambda_i\}$ denote the eigenvalues of the Jacobian $\mathbf{D}_{\boldsymbol{\alpha}} \mathbf{f}(\boldsymbol{\alpha}^*)$. If $\max_i |\lambda_i| < 1$, then $\boldsymbol{\alpha}^*$ is stable; if $\max_i |\lambda_i| > 1$, then $\boldsymbol{\alpha}^*$ is unstable. We focus on this unique stable equilibrium in our numerical illustrations.

Why do we focus on the unique stable equilibrium? First, consistent with Samuelson's (1947) correspondence principle, the single stable equilibrium has local comparative statics that comport with common sense economic intuition. By contrast, the unstable equilibria feature comparative statics that conflict with standard intuition.[4] To understand the intuition for this result, consider the impact of some parameter γ on the equilibrium. An equilibrium satisfies $\boldsymbol{\alpha}^* = \mathbf{f}(\boldsymbol{\alpha}^*, \gamma)$. By the implicit function theorem, we have

$$\mathbf{D}_\gamma \boldsymbol{\alpha}^* = [\mathbf{I} - \mathbf{D}_{\boldsymbol{\alpha}} \mathbf{f}(\boldsymbol{\alpha}^*, \gamma)]^{-1} \mathbf{D}_\gamma \mathbf{f}(\boldsymbol{\alpha}^*, \gamma).$$

If an equilibrium is stable (as well as isolated and non-degenerate), then all of the eigenvalues of $\mathbf{D}_{\boldsymbol{\alpha}} \mathbf{f}(\boldsymbol{\alpha}^*, \gamma)$ have a modulus less than 1 and we can write

$$\mathbf{D}_\gamma \boldsymbol{\alpha}^* = [\textstyle\sum_{i=0}^{\infty} (\mathbf{D}_{\boldsymbol{\alpha}} \mathbf{f}(\boldsymbol{\alpha}^*, \gamma))^i] \mathbf{D}_\gamma \mathbf{f}(\alpha^*, \gamma).$$

This says that comparative statics on $\boldsymbol{\alpha}^*$ have a straightforward interpretation in terms of a dynamic adjustment process. The first-round direct effect is $\mathbf{D}_\gamma \mathbf{f}(\boldsymbol{\alpha}^*, \gamma)$. The second-round indirect effect is then $\mathbf{D}_{\boldsymbol{\alpha}} \mathbf{f}(\boldsymbol{\alpha}^*, \gamma) \mathbf{D}_\gamma \mathbf{f}(\boldsymbol{\alpha}^*, \gamma)$. The third-round indirect effect is $(\mathbf{D}_{\boldsymbol{\alpha}} \mathbf{f}(\boldsymbol{\alpha}^*, \gamma))^2 \mathbf{D}_\gamma \mathbf{f}(\boldsymbol{\alpha}^*, \gamma)$. The total effect is the sum across all rounds.[5] Samuelson's correspondence principle refers to this correspondence between equilibrium comparative statics and the result of this dynamic adjustment process.

When equilibrium only involves a single variable, knowledge that an equilibrium is stable (unstable) allows one to unambiguously determine the sign of equilibrium comparative statics (Samuelson (1947)).[6] Things are more complicated when an equilibrium involves multiple unknowns as it does in our general (Arrow and Hahn (1971) and Echenique (2002, 2008)). In multivariate settings, knowledge that an equilibrium is stable (or unstable) only allows one to unambiguously sign equilibrium comparative statics in very special cases. However, the fact that an equilibrium is stable, still has qualitative implications for comparative statics.

Second, the unique stable equilibrium of our general model has a well-behaved limit as investors' risk tolerance grows large ($\tau \to \infty$) in the sense that it converges to the equilibrium with risk-neutral investors. By contrast, the unstable equilibria explode in this limit with one or more price-impact coefficients going to infinity. Similarly, as the volatility of supply shocks vanishes, the stable equilibrium converges to the equilibrium with deterministic supply. Again, the unstable equilibria explode in this limit.

[4]For instance, in the simple case discussed above with only a single risky asset, the low price-impact equilibrium is stable and the high price-impact equilibrium is unstable. At the stable equilibrium, an increase in the volatility of short-term rate shocks or the volatility of supply shocks is associated with an increase in the price-impact coefficient and an increase in the volatility of returns. By contrast, these comparitive statics take the opposite sign at the unstable equilibrium.

[5]By contrast, if an equilibrium is unstable then some of the eigenvalues of $\mathbf{D}_{\boldsymbol{\alpha}} \mathbf{f}(\boldsymbol{\alpha}^*, \gamma)$ have a modulus greater than 1. Thus, we have $[\mathbf{I} - \mathbf{D}_{\boldsymbol{\alpha}} \mathbf{f}(\boldsymbol{\alpha}^*, \gamma)]^{-1} \neq [\sum_{i=0}^{\infty} (\mathbf{D}_{\boldsymbol{\alpha}} \mathbf{f}(\boldsymbol{\alpha}^*, \gamma))^i]$ so $\mathbf{D}_\gamma \alpha^* \neq [\sum_{i=0}^{\infty} (\mathbf{D}_{\boldsymbol{\alpha}} \mathbf{f}(\boldsymbol{\alpha}^*, \gamma))^i] \mathbf{D}_\gamma \mathbf{f}(\alpha^*, \gamma)$ and comparative statics don't have this intuitive interptation.

[6]Consider a univariate fixed point problem of the form $\alpha^* = f(\alpha^*, \gamma)$. An equilibrium is stable if $|\partial f(\alpha^*, \gamma) / \partial \alpha| < 1$. The comparative static with respect to γ is $\partial \alpha^* / \partial \gamma = [1 - \partial f(\alpha^*, \gamma) / \partial \alpha]^{-1} [\partial f(\alpha^*, \gamma) / \partial \gamma]$. When $|\partial f(\alpha^*, \gamma) / \partial \alpha| < 1$, we have $\partial \alpha^* / \partial \gamma = [\sum_{i=0}^{\infty} (\partial f(\alpha^*, \gamma) / \partial \alpha)^i] [\partial f(\alpha^*, \gamma) / \partial \gamma] \propto \partial f(\alpha^*, \gamma) / \partial \gamma$. However, when $\partial f(\alpha^*, \gamma) / \partial \alpha > 1$, $\partial \alpha^* / \partial \gamma \propto -\partial f(\alpha^*, \gamma) / \partial \gamma$.

A.4 Equilibrium solution

To find a rational expectations equilibrium of our overlapping-generations model, we need to set up the fixed point problem discussed above. In our baseline model, one can either think of this as a fixed point problem involving $\mathbf{A}$ or $\mathbf{V}$.

To set up this problem, we begin with the market-clearing conditions $E_t[\mathbf{rx}_{t+1}] = \tau^{-1}\mathbf{V}\mathbf{s}_t$. We write the vector of excess returns as

$$\mathbf{rx}_{t+1} \equiv \begin{bmatrix} rx^{y}_{t+1} \\ rx^{y*}_{t+1} \\ rx^{q}_{t+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{1-\delta}y_t - \frac{\delta}{1-\delta}y_{t+1} - i_t \\ \frac{1}{1-\delta}y^*_t - \frac{\delta}{1-\delta}y^*_{t+1} - i^*_t \\ q_{t+1} - q_t + (i^*_t - i_t) \end{bmatrix} = \mathbf{B}_0\mathbf{y}_t + \mathbf{B}_1\mathbf{y}_{t+1} + \mathbf{R}_1\mathbf{z}_t + \mathbf{r}_0,$$

where

$$\mathbf{B}_0 \equiv \begin{bmatrix} \frac{1}{1-\delta} & 0 & 0 \\ 0 & \frac{1}{1-\delta} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \mathbf{B}_1 \equiv \begin{bmatrix} -\frac{\delta}{1-\delta} & 0 & 0 \\ 0 & -\frac{\delta}{1-\delta} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{R}_1 \equiv \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \end{bmatrix}, \text{ and } \mathbf{r}_0 \equiv \begin{bmatrix} -\bar{\imath} \\ -\bar{\imath} \\ 0 \end{bmatrix}.$$

Substituing $\mathbf{y}_t = \mathbf{a} + \mathbf{A}\mathbf{z}_t$ and $\mathbf{y}_{t+1} = \mathbf{a} + \mathbf{A}\mathbf{z}_{t+1} = \mathbf{a} + \mathbf{A}\mathbf{\Phi}\mathbf{z}_t + \mathbf{A}\boldsymbol{\varepsilon}_{t+1}$, we obtain

$$\mathbf{rx}_{t+1} = [\mathbf{B}_0\mathbf{a} + \mathbf{B}_1\mathbf{a} + \mathbf{r}_0] + [\mathbf{B}_0\mathbf{A} + \mathbf{B}_1\mathbf{A}\mathbf{\Phi} + \mathbf{R}_1]\mathbf{z}_t + [\mathbf{B}_1\mathbf{A}]\boldsymbol{\varepsilon}_{t+1},$$

which implies

$$E_t[\mathbf{rx}_{t+1}] = [\mathbf{B}_0\mathbf{a} + \mathbf{B}_1\mathbf{a} + \mathbf{r}_0] + [\mathbf{B}_0\mathbf{A} + \mathbf{B}_1\mathbf{A}\mathbf{\Phi} + \mathbf{R}_1]\mathbf{z}_t,$$

and

$$\mathbf{V} \equiv Var_t[\mathbf{rx}_{t+1}] = \mathbf{B}_1\mathbf{A}\mathbf{\Sigma}\mathbf{A}'\mathbf{B}_1'.$$

Finally, the vector of net asset supplies that fixed-income investor must hold in equilibrium can be written as $\mathbf{s}_t = \mathbf{s}_0 + \mathbf{S}_1\mathbf{z}_t$, where

$$\mathbf{S}_1 \equiv \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } \mathbf{s}_0 \equiv \begin{bmatrix} \bar{s}^y \\ \bar{s}^y \\ 0 \end{bmatrix}.$$

Thus, the market clearing conditions $E_t[\mathbf{rx}_{t+1}] = \tau^{-1}\mathbf{V}\mathbf{s}_t$ can be written as

$$[\mathbf{B}_0\mathbf{a} + \mathbf{B}_1\mathbf{a} + \mathbf{r}_0] + [\mathbf{B}_0\mathbf{A} + \mathbf{B}_1\mathbf{A}\mathbf{\Phi} + \mathbf{R}_1]\mathbf{z}_t = \tau^{-1}(\mathbf{B}_1\mathbf{A}\mathbf{\Sigma}\mathbf{A}'\mathbf{B}_1')(\mathbf{s}_0 + \mathbf{S}_1\mathbf{z}_t). \tag{4}$$

Since $\mathbf{B}_0$, $\mathbf{B}_1$, and $\mathbf{\Phi}$ are diagonal, it follows that $[\mathbf{B}_0\mathbf{A} + \mathbf{B}_1\mathbf{A}\mathbf{\Phi}] = \mathbf{A} \circ [\mathbf{B}_0\mathbf{E} + \mathbf{B}_1\mathbf{E}\mathbf{\Phi}]$ where $\circ$ denotes element-wise matrix multiplication (i.e., the Hadamard product) and $\mathbf{E}$ is a 3×5 matrix of 1s. Specifically, we have

$$[\mathbf{B}_0\mathbf{E} + \mathbf{B}_1\mathbf{E}\mathbf{\Phi}] = \begin{bmatrix} \frac{1-\delta\phi_i}{1-\delta} & \frac{1-\delta\phi_i}{1-\delta} & \frac{1-\delta\phi_{sy}}{1-\delta} & \frac{1-\delta\phi_{sy}}{1-\delta} & \frac{1-\delta\phi_{sq}}{1-\delta} \\ \frac{1-\delta\phi_i}{1-\delta} & \frac{1-\delta\phi_i}{1-\delta} & \frac{1-\delta\phi_{sy}}{1-\delta} & \frac{1-\delta\phi_{sy}}{1-\delta} & \frac{1-\delta\phi_{sq}}{1-\delta} \\ \phi_i - 1 & \phi_i - 1 & \phi_{s^y} - 1 & \phi_{s^y} - 1 & \phi_{s^q} - 1 \end{bmatrix}.$$

Thus, matching the matrices multiplying the state vector $\mathbf{z}_t$, we see that $\mathbf{A}$ must solve the following fixed point problem:

$$\mathbf{A} = \left[\tau^{-1}\mathbf{B}_1\mathbf{A}\mathbf{\Sigma}\mathbf{A}'\mathbf{B}_1'\mathbf{S}_1 - \mathbf{R}_1\right] \oslash [\mathbf{B}_0\mathbf{E} + \mathbf{B}_1\mathbf{E}\mathbf{\Phi}], \tag{5}$$

where $\oslash$ denotes element-wise matrix division (i.e., Hadamard division).[7] As we show below, the 6

[7] The price function that will clear markets today when agents conjecture that the price function will be $\mathbf{A}$ at all

elements for short rates are trivially pinned down, so this can be reduced to a fixed point problem involving the 9 price impact coefficients which govern how prices respond to changes in asset supply. Matching coefficients on the vector of constants, we find

$$(\mathbf{B}_0 + \mathbf{B}_1)\,\mathbf{a} = \left[\tau^{-1}\mathbf{B}_1\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}'\mathbf{B}_1'\mathbf{s}_0 - \mathbf{r}_0\right]. \tag{6}$$

Since the final row of $\mathbf{B}_0 + \mathbf{B}_1$ only contains zeros, the constants for the two bond yields are pinned down in equilibrium, but the constant for the exchange rate is not pinned down.

Coefficients on short rates To further characterize the solution $\mathbf{A}$, we partition $\mathbf{z}_t$ as $\mathbf{z}_t = [\mathbf{z}_{i,t}', \mathbf{z}_{s,t}']'$ where $\mathbf{z}_{i,t} = \left[i_t - \bar{\imath},\ \ i_t^* - \bar{\imath}\right]'$ and $\mathbf{z}_{s,t} = \left[s_t^y - \bar{s}^y,\ \ s_t^{y*} - \bar{s}^y,\ \ s_t^q\right]'$. Thus, $\mathbf{z}_{i,t}$ contains the two state variables that drive short rates and $\mathbf{z}_{s,t}$ contains the three state variables that drive asset supply. Similarly, we partition $\mathbf{A}$ as $\mathbf{A} = [\mathbf{A}_i,\ \mathbf{A}_s]$ where $\mathbf{A}_i$ is the 3×2 matrix of loadings on $\mathbf{x}_{i,t}$ and $\mathbf{A}_s$ is the 3×3 matrix of loadings on $\mathbf{x}_{s,t}$. For an arbitrary matrix $\mathbf{X}$, let $\mathbf{X}^{[n-m]}$ for $m > n$ be the submatrix consisting of columns n, $n+1$, ..., $m-1$, m of $\mathbf{X}$. Given the form of $\mathbf{R}_1$ and $\mathbf{S}_1$, we see that

$$\mathbf{A}_i = -\mathbf{R}_1^{[1-2]} \oslash [\mathbf{B}_0\mathbf{E} + \mathbf{B}_1\mathbf{E}\boldsymbol{\Phi}]^{[1-2]} = -\begin{bmatrix} -1 & 0 \\ 0 & -1 \\ -1 & 1 \end{bmatrix} \oslash \begin{bmatrix} \frac{1-\delta\phi_i}{1-\delta} & \frac{1-\delta\phi_i}{1-\delta} \\ \frac{1-\delta\phi_i}{1-\delta} & \frac{1-\delta\phi_i}{1-\delta} \\ \phi_i - 1 & \phi_i - 1 \end{bmatrix} = \begin{bmatrix} \frac{1-\delta}{1-\delta\phi_i} & 0 \\ 0 & \frac{1-\delta}{1-\delta\phi_i} \\ -\frac{1}{1-\phi_i} & \frac{1}{1-\phi_i} \end{bmatrix}. \tag{7}$$

Thus, the coefficients $\mathbf{A}_i$ governing how asset prices respond to interest rates are trivially pinned down in any rational expectations equilibrium.

Price impact coefficients Next, making use of the assumed orthogonality between short rate shocks and supply shocks, we partition the variance-covariance matrix of the shocks as

$$\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_i & \mathbf{0}_{2\times 3} \\ \mathbf{0}_{3\times 2} & \boldsymbol{\Sigma}_s \end{bmatrix} \text{ where } \boldsymbol{\Sigma}_i = \begin{bmatrix} \sigma_i^2 & \rho\sigma_i^2 \\ \rho\sigma_i^2 & \sigma_i^2 \end{bmatrix} \text{ and } \boldsymbol{\Sigma}_s = \begin{bmatrix} \sigma_{s^y}^2 & 0 & 0 \\ 0 & \sigma_{s^y}^2 & 0 \\ 0 & 0 & \sigma_{s^q}^2 \end{bmatrix}.$$

Thus, the variance covariance matrix of excess returns becomes

$$\mathbf{V} = (\mathbf{B}_1\mathbf{A}_i\boldsymbol{\Sigma}_i\mathbf{A}_i'\mathbf{B}_1') + (\mathbf{B}_1\mathbf{A}_s\boldsymbol{\Sigma}_s\mathbf{A}_s'\mathbf{B}_1'). \tag{8}$$

In other words, $\mathbf{V}$ is the sum of a term $(\mathbf{B}_1\mathbf{A}_i\boldsymbol{\Sigma}_i\mathbf{A}_i'\mathbf{B}_1')$ reflecting the fundamental risk generated by future shocks to short rates and a term $(\mathbf{B}_1\mathbf{A}_s\boldsymbol{\Sigma}_s\mathbf{A}_s'\mathbf{B}_1')$ reflecting the non-fundamental risk generated by future shocks to asset supply. Again, making use of the form of $\mathbf{R}_1$ and $\mathbf{S}_1$, we obtain the following fixed point problem involving $\mathbf{A}_s$ alone:

$$\mathbf{A}_s = \mathbf{F}_s(\mathbf{A}_s) \equiv \tau^{-1}\left[(\mathbf{B}_1\mathbf{A}_i\boldsymbol{\Sigma}_i\mathbf{A}_i'\mathbf{B}_1') + (\mathbf{B}_1\mathbf{A}_s\boldsymbol{\Sigma}_s\mathbf{A}_s'\mathbf{B}_1')\right] \oslash [\mathbf{B}_0\mathbf{E} + \mathbf{B}_1\mathbf{E}\boldsymbol{\Phi}]^{[3-5]}. \tag{9}$$

As discussed above, the operator $\mathbf{F}_s(\mathbf{A}_s)$ gives the price function $\mathbf{y}_t = \mathbf{g}(\mathbf{A}_s) + \mathbf{A}_i\mathbf{z}_{i,t} + \mathbf{F}_s(\mathbf{A}_s)\,\mathbf{z}_{s,t}$ that will clear the markets for long-term bonds and FX when agents conjecture that the risk of holding of assets is determined by the price function $\mathbf{y}_{t+1} = \mathbf{a}_0 + \mathbf{A}_i\mathbf{z}_{i,t+1} + \mathbf{A}_s\mathbf{z}_{s,t+1}$. In other words, equation

future dates is $\mathbf{A} = \mathbf{B}_0^{-1}\left[\tau^{-1}\left(\mathbf{B}_1\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}'\mathbf{B}_1'\right)\mathbf{S}_1 - (\mathbf{B}_1\mathbf{A}\boldsymbol{\Phi} + \mathbf{R}_1)\right]$. Of course, any solution to this modified fixed point problem is a solution to the fixed point problem in equation (5) and vice versa.

(9) says that the equilibrium price impact coefficient must satisfy

$$\begin{bmatrix} \alpha^{y}_{s^y} & \alpha^{y}_{s^{y*}} & \alpha^{y}_{s^q} \\ \alpha^{y*}_{s^y} & \alpha^{y*}_{s^{y*}} & \alpha^{y*}_{s^q} \\ \alpha^{q}_{s^y} & \alpha^{q}_{s^{y*}} & \alpha^{q}_{s^q} \end{bmatrix} = \tau^{-1} \begin{bmatrix} \frac{1-\delta}{1-\delta\phi_{s^y}} V_y & \frac{1-\delta}{1-\delta\phi_{s^y}} C_{y,y*} & \frac{1-\delta}{1-\delta\phi_{s^q}} C_{y,q} \\ \frac{1-\delta}{1-\delta\phi_{s^y}} C_{y,y*} & \frac{1-\delta}{1-\delta\phi_{s^y}} V_{y*} & \frac{1-\delta}{1-\delta\phi_{s^q}} C_{y*,q} \\ -\frac{1}{1-\phi_{s^y}} C_{y,q} & -\frac{1}{1-\phi_{s^y}} C_{y*,q} & -\frac{1}{1-\phi_{s^q}} V_q \end{bmatrix}, \tag{10}$$

where $V_a \equiv Var\left[rx^a_{t+1}\right]$ and $C_{a,a'} \equiv Cov[rx^a_{t+1}, rx^{a'}_{t+1}]$ are the equilibrium return (co)variances.

The variance-covariance matrix in the absence of supply risk is $(\mathbf{B}_1 \mathbf{A}_i \mathbf{\Sigma}_i \mathbf{A}_i' \mathbf{B}_1') =$

$$\begin{bmatrix} \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \sigma_i^2 & \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \rho\sigma_i^2 & \frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2(1-\rho) \\ \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \rho\sigma_i^2 & \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \sigma_i^2 & -\frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2(1-\rho) \\ \frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2(1-\rho) & -\frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2(1-\rho) & \left(\frac{1}{1-\phi_i}\right)^2 2\sigma_i^2(1-\rho) \end{bmatrix}. \tag{11}$$

The contribution of supply risk to the variance-covariance matrix is $(\mathbf{B}_1 \mathbf{A}_s \mathbf{\Sigma}_s \mathbf{A}_s{}' \mathbf{B}_1') =$

$$\begin{bmatrix} \left(\frac{\delta}{1-\delta}\right)^2 \begin{pmatrix} (\alpha^y_{s^y})^2 \sigma^2_{s^y} \\ + (\alpha^y_{s^{y*}})^2 \sigma^2_{s^y} \\ + (\alpha^y_{s^q})^2 \sigma^2_{s^q} \end{pmatrix} & \left(\frac{\delta}{1-\delta}\right)^2 \begin{pmatrix} (\alpha^y_{s^y}\alpha^{y*}_{s^y}) \sigma^2_{s^y} \\ + (\alpha^y_{s^{y*}}\alpha^{y*}_{s^{y*}}) \sigma^2_{s^y} \\ + (\alpha^y_{s^q}\alpha^{y*}_{s^q}) \sigma^2_{s^q} \end{pmatrix} & -\left(\frac{\delta}{1-\delta}\right) \begin{pmatrix} (\alpha^y_{s^y}\alpha^q_{s^y}) \sigma^2_{s^y} \\ + (\alpha^y_{s^{y*}}\alpha^q_{s^{y*}}) \sigma^2_{s^y} \\ + (\alpha^y_{s^q}\alpha^q_{s^q}) \sigma^2_{s^q} \end{pmatrix} \\ \left(\frac{\delta}{1-\delta}\right)^2 \begin{pmatrix} (\alpha^y_{s^y}\alpha^{y*}_{s^y}) \sigma^2_{s^y} \\ + (\alpha^y_{s^{y*}}\alpha^{y*}_{s^{y*}}) \sigma^2_{s^y} \\ + (\alpha^y_{s^q}\alpha^{y*}_{s^q}) \sigma^2_{s^q} \end{pmatrix} & \left(\frac{\delta}{1-\delta}\right)^2 \begin{pmatrix} (\alpha^{y*}_{s^y})^2 \sigma^2_{s^y} \\ + (\alpha^{y*}_{s^{y*}})^2 \sigma^2_{s^y} \\ + (\alpha^{y*}_{s^q})^2 \sigma^2_{s^q} \end{pmatrix} & -\left(\frac{\delta}{1-\delta}\right) \begin{pmatrix} (\alpha^{y*}_{s^y}\alpha^q_{s^y}) \sigma^2_{s^y} \\ + (\alpha^{y*}_{s^{y*}}\alpha^{y*}_{s^{y*}}) \sigma^2_{s^y} \\ + (\alpha^{y*}_{s^q}\alpha^q_{s^q}) \sigma^2_{s^q} \end{pmatrix} \\ -\left(\frac{\delta}{1-\delta}\right) \begin{pmatrix} (\alpha^y_{s^y}\alpha^q_{s^y}) \sigma^2_{s^y} \\ + (\alpha^y_{s^{y*}}\alpha^q_{s^{y*}}) \sigma^2_{s^y} \\ + (\alpha^y_{s^q}\alpha^q_{s^q}) \sigma^2_{s^q} \end{pmatrix} & -\left(\frac{\delta}{1-\delta}\right) \begin{pmatrix} (\alpha^{y*}_{s^y}\alpha^q_{s^y}) \sigma^2_{s^y} \\ + (\alpha^{y*}_{s^{y*}}\alpha^q_{s^{y*}}) \sigma^2_{s^y} \\ + (\alpha^{y*}_{s^q}\alpha^q_{s^q}) \sigma^2_{s^q} \end{pmatrix} & \begin{pmatrix} (\alpha^q_{s^y})^2 \sigma^2_{s^y} \\ + (\alpha^q_{s^{y*}})^2 \sigma^2_{s^y} \\ + (\alpha^q_{s^q})^2 \sigma^2_{s^q} \end{pmatrix} \end{bmatrix}. \tag{12}$$

As we will show below, any solution to this fixed point problem must satisfy $\alpha^y_{s^y} = \alpha^{y*}_{s^{y*}}$, $\alpha^y_{s^{y*}} = \alpha^{y*}_{s^y}$, $\alpha^y_{s^q} = -\alpha^{y*}_{s^q}$, and $\alpha^q_{s^y} = -\alpha^q_{s^{y*}}$.

Recasting the fixed-point problem in terms of the variance-covariance matrix We now recast this fixed point problem involving the 3×3 matrix $\mathbf{A}_s$ as a fixed point problem involving the 3×3 variance-covariance matrix of returns $\mathbf{V}$—i.e., a fixed point of the form $\mathbf{V} = \mathbf{G}(\mathbf{V})$. While $\mathbf{A}_s$ is not symmetric, $\mathbf{V}$ is symmetric, effectively reducing the fixed-point in 9 unknowns to one involving 6 unknowns. Specifically, making use of equations (8), (10), (11), and (12) and defining the constants,

$$g_y \equiv \tau^{-1}\frac{\delta}{1-\delta\phi_{s^y}}\sigma_{s^y},\ g_q \equiv \tau^{-1}\frac{\delta}{1-\delta\phi_{s^q}}\sigma_{s^q},\ h_y \equiv \tau^{-1}\frac{1}{1-\phi_{s^y}}\sigma_{s^y},\text{ and } h_q \equiv \tau^{-1}\frac{1}{1-\phi_{s^q}}\sigma_{s^q}, \tag{13}$$

we find that $\mathbf{V}$ must satisfy the following system of 6 equations in 6 unknowns:

$$V_y = \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \sigma_i^2 + \left[g_y^2 (V_y)^2 + g_y^2 (C_{y,y^*})^2 + g_q^2 (C_{y,q})^2\right] \tag{14a}$$

$$V_{y^*} = \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \sigma_i^2 + \left[g_y^2 (V_{y^*})^2 + g_y^2 (C_{y,y^*})^2 + g_q^2 (C_{y^*,q})^2\right] \tag{14b}$$

$$V_q = \left(\frac{1}{1-\phi_i}\right)^2 2\sigma_i^2 (1-\rho) + \left[h_y^2 (C_{y,q})^2 + h_y^2 (C_{y^*,q})^2 + h_q^2 (V_q)^2\right] \tag{14c}$$

$$C_{y,y*} = \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \rho\sigma_i^2 + \left[g_y^2 V_y C_{y,y^*} + g_y^2 V_{y*} C_{y,y^*} + g_q^2 C_{y,q} C_{y*,q}\right] \tag{14d}$$

$$C_{y,q} = \frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2 (1-\rho) + \left[g_y h_y V_y C_{y,q} + g_y h_y C_{y,y^*} C_{y*,q} + g_q h_q C_{y,q} V_q\right] \tag{14e}$$

$$C_{y*,q} = -\frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2 (1-\rho) + \left[g_y h_y V_{y^*} C_{y^*,q} + g_y h_y C_{y,y^*} C_{y,q} + g_q h_q C_{y^*,q} V_q\right] \tag{14f}$$

These equations give the actual risk of holding assets when agents make specific conjectures about future asset risk and thus demand commensurate discounts to absorb supply-and-demand shocks.

It is easy to see that any solution to this fixed-point problem must satisfy $V_y = V_{y*}$ and $C_{y,q} = -C_{y*,q}$. To see this, subtract the second equation from the first to obtain $V_y - V_{y^*} = g_q^2 \times ((C_{y,q})^2 - (C_{y^*,q})^2)$. Next, add the fifth and sixth equations to obtain $C_{y,q} + C_{y*,q} = [g_y h_y C_{y^*,q} \times (V_{y^*} - V_y)] \div [1 - g_y h_y (C_{y,y^*} + V_y) - g_q h_q V_q]$. Combining these two expressions, we have

$$C_{y,q} + C_{y*,q} = \overbrace{\left[\frac{g_y h_y g_q^2 C_{y^*,q} (C_{y,q} - C_{y*,q})}{1 - g_y h_y (C_{y,y^*} + V_y) - g_q h_q V_q}\right]}^{\neq 0} \times (C_{y,q} + C_{y*,q}),$$

which implies that $C_{y,q} + C_{y*,q} = 0$. It then follows that $V_y = V_{y*}$.

Imposing this symmetry condition, we are left with a fixed point problem involving just four unknowns:

$$V_y = \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \sigma_i^2 + \left[g_y^2 (V_y)^2 + g_y^2 (C_{y,y^*})^2 + g_q^2 (C_{y,q})^2\right] \tag{15a}$$

$$V_q = \left(\frac{1}{1-\phi_i}\right)^2 2\sigma_i^2 (1-\rho) + \left[2h_y^2 (C_{y,q})^2 + h_q^2 (V_q)^2\right] \tag{15b}$$

$$C_{y,y^*} = \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \rho\sigma_i^2 + \left[2g_y^2 V_y C_{y,y^*} - g_q^2 (C_{y,q})^2\right] \tag{15c}$$

$$C_{y,q} = \frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2 (1-\rho) + \left[h_y g_y (V_y - C_{y,y^*}) + h_q g_q V_q\right] C_{y,q} \tag{15d}$$

A.5 Characterizing the solution

We now characterize the solution to the system of equations in (15). We first discuss the solution in the limiting case where supply risk vanishes. We then discuss the solution in the general case where both $\sigma_{s_y}^2 > 0$ and $\sigma_{s_q}^2 > 0$. Finally, we discuss the solution in the special case where $\sigma_{s_y}^2 > 0$ and $\sigma_{s_q}^2 = 0$.

A.5.1 Limiting case with no supply risk

Taking the limit as supply risk grows small ($\mathbf{\Sigma}_s \to \mathbf{0}$), we have

$$\lim_{\mathbf{\Sigma}_s \to \mathbf{0}} \mathbf{V} = (\mathbf{B}_1 \mathbf{A}_i \mathbf{\Sigma}_i \mathbf{A}_i' \mathbf{B}_1') = \begin{bmatrix} \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \sigma_i^2 & \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \rho\sigma_i^2 & \frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2(1-\rho) \\ \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \rho\sigma_i^2 & \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \sigma_i^2 & -\frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2(1-\rho) \\ \frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2(1-\rho) & -\frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2(1-\rho) & \left(\frac{1}{1-\phi_i}\right)^2 2\sigma_i^2(1-\rho) \end{bmatrix},$$

and $\lim_{\mathbf{\Sigma}_s \to \mathbf{0}} \mathbf{A}_s = \tau^{-1}(\mathbf{B}_1 \mathbf{A}_i \mathbf{\Sigma}_i \mathbf{A}_i' \mathbf{B}_1') \oslash [\mathbf{B}_0 \mathbf{E} + \mathbf{B}_1 \mathbf{E} \mathbf{\Phi}]^{[3-5]}$

$$= \tau^{-1} \begin{bmatrix} \frac{1-\delta}{1-\delta\phi_{sy}}\left(\frac{\delta}{1-\delta\phi_i}\right)^2 \sigma_i^2 & \frac{1-\delta}{1-\delta\phi_{sy}}\left(\frac{\delta}{1-\delta\phi_i}\right)^2 \rho\sigma_i^2 & \frac{1-\delta}{1-\delta\phi_{sq}}\frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2(1-\rho) \\ \frac{1-\delta}{1-\delta\phi_{sy}}\left(\frac{\delta}{1-\delta\phi_i}\right)^2 \rho\sigma_i^2 & \frac{1-\delta}{1-\delta\phi_{sy}}\left(\frac{\delta}{1-\delta\phi_i}\right)^2 \sigma_i^2 & -\frac{1-\delta}{1-\delta\phi_{sq}}\frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2(1-\rho) \\ -\frac{1}{1-\phi_{sy}}\frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2(1-\rho) & \frac{1}{1-\phi_{sy}}\frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2(1-\rho) & -\frac{1}{1-\phi_{sq}}\left(\frac{1}{1-\phi_i}\right)^2 2\sigma_i^2(1-\rho) \end{bmatrix}.$$

In the limit where supply risk grows small ($\mathbf{\Sigma}_s \to \mathbf{0}$), all of the price-impact coefficients have the expected signs. Since the model's stable equilibrium is continuous in the model's underlying parameters, this guarantees that the price-impact coefficients will always have same signs when supply risk is small.

A.5.2 General solution with both bond and FX supply shocks

When $\sigma_{s^y}^2 > 0$ and $\sigma_{s^q}^2 > 0$, solving the model involves reducing a system of four quadratic equations in four unknowns to an equation that behaves like a cubic in a single unknown, characterizing solutions to that equation, and then solving the rest of the system. We assume throughout that $\rho < 1$.

Step #1: Solve for $\Delta \equiv V_y - C_{y,y*}$ as a function of $C_{y,q}$. We subtract condition (15c) for $C_{y,y*}$ from condition (15a) for V_y to obtain the following quadratic in $\Delta \equiv V_y - C_{y,y*}$:

$$(V_y - C_{y,y*}) = \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \sigma_i^2(1-\rho) + g_y^2 (V_y - C_{y,y*})^2 + 2g_q^2 (C_{y,q})^2.$$

Since the right-hand-side is always positive, we must have $\Delta = V_y - C_{y,y*} > 0$. We then solve the following quadratic equation for $\Delta \equiv (V_y - C_{y,y*})$ as a function of $C_{y,q}$:

$$0 = \overbrace{g_y^2}^{a_\Delta > 0} \Delta^2 - \Delta + \overbrace{\left[2g_q^2 (C_{y,q})^2 + \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \sigma_i^2(1-\rho)\right]}^{c_\Delta(C_{y,q})>0}.$$

A real solution only exists if

$$\begin{aligned} 1 \quad &> \quad 4g_y^2 \left[2g_q^2 (C_{y,q})^2 + \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \sigma_i^2(1-\rho)\right] \\ &= \quad 4\left(\frac{\delta\tau^{-1}\sigma_{s^y}}{1-\delta\phi_{s^y}}\right)^2 \left[2\left(\frac{\delta\tau^{-1}\sigma_{s^q}}{1-\delta\phi_{s^q}}\right)^2 (C_{y,q})^2 + \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \sigma_i^2(1-\rho)\right] \end{aligned}$$

or $\tau > \hat{\tau}_\Delta (C_{y,q})$—i.e., if risk tolerance is sufficiently larger—where this cutoff is an increasing function of $C_{y,q}$.

Let $\Delta (C_{y,q})$ denote the smaller solution to this quadratic:

$$\Delta (C_{y,q}) = \frac{1-\sqrt{1-4a_\Delta c_\Delta (C_{y,q})}}{2a_\Delta}, \tag{16}$$

which corresponds to the stable solution to the fixed point problem. Call the solution $\Delta (C_{y,q}) > 0$ and note that

$$\Delta' (C_{y,q}) = \frac{2g_q^2 C_{y,q}}{1-g_y^2 2\Delta (C_{y,q})} \propto C_{y,q},$$

since $1-g_y^2 2\Delta (C_{y,q}) > 0$ at the relevant solution. Thus, $\Delta' (0) = 0$, $\Delta' (C_{y,q}) > 0$ when $C_{y,q} > 0$, and $\Delta' (C_{y,q}) < 0$ when $C_{y,q} < 0$. Also note that

$$\Delta'' (C_{y,q}) = \frac{2g_q^2 \left(1-g_y^2 2\Delta (C_{y,q})\right) + 4g_y^2 g_q^2 \Delta' (C_{y,q}) C_{y,q}}{\left[1-(g_y)^2 2\Delta^* (C_{y,q})\right]^2} > 0.$$

Thus, $\Delta (C_{y,q})$ is a positive, U-shaped function of $C_{y,q}$. Also, note that we have $g_y^2 \Delta < 1$.

Step #2: Solve for V_q as a function of $C_{y,q}$. Next, we solve for V_q as a function of $C_{y,q}$. Rearranging condition (15b) for V_q, we want to solve the following quadratic for V_q as a function of $C_{y,q}$:

$$0 = \overbrace{h_q^2}^{a_{V_q}>0} (V_q)^2 - V_q + \overbrace{\left(2h_y^2 (C_{y,q})^2 + \left(\frac{1}{1-\phi_i}\right)^2 2\sigma_i^2 (1-\rho)\right)}^{c_{V_q}(C_{y,q})>0}.$$

A real solution only exists if $\tau > \hat{\tau}_q (C_{y,q})$ where this cutoff is an increasing function of $C_{y,q}$. Let

$$\nu_q (C_{y,q}) = \frac{1-\sqrt{1-4a_{V_q} c_{V_q} (C_{y,q})}}{2a_{V_q}} > 0 \tag{17}$$

denote the smaller root of this quadratic which corresponds to the stable solution to the fixed point problem. Note that, at the relevant solution to this quadratic, $\nu_q' (0) = 0$, $\nu_q' (C_{y,q}) > 0$ when $C_{y,q} > 0$, and $\nu_q' (C_{y,q}) < 0$ when $C_{y,q} < 0$. Also, note that $\nu_q'' (C_{y,q}) > 0$. Thus, $\nu_q (C_{y,q})$ is also a positive, U-shaped function of $C_{y,q}$. Also, note that we have $h_q^2 v_q < 1$.

Step #3: Plug these functions back into condition (15d) for $C_{y,q}$ and solve for $C_{y,q}$. Plugging in these two functions—$\Delta (C_{y,q})$ and $\nu_q (C_{y,q})$—into condition (15d) for $C_{y,q}$, we obtain the following equation in one unknown for $C_{y,q}$:

$$C_{y,q} = F (C_{y,q}) \equiv \frac{\delta}{1-\delta\phi_i} \frac{1}{1-\phi_i} \sigma_i^2 (1-\rho) + \left[h_y g_y \Delta (C_{y,q}) + h_q g_q \nu_q (C_{y,q})\right] C_{y,q}. \tag{18}$$

We now use the properties of $F (C_{y,q})$ to characterize the solutions to $C_{y,q} = F (C_{y,q})$. Specifically, $F (C_{y,q})$ has the following properties:

- $F (0) = \frac{\delta}{1-\delta\phi_i} \frac{1}{1-\phi_i} \sigma_i^2 (1-\rho) > 0$.

- $F'(C_{y,q}) > 0$ for all $C_{y,q}$. To see this note that

$$F'(C_{y,q}) = [h_y g_y \Delta(C_{y,q}) + h_q g_q \nu_q(C_{y,q})] + [h_y g_y \Delta'(C_{y,q}) + h_q g_q \nu'_q(C_{y,q})] C_{y,q}$$

  Since $\Delta(C_{y,q}) > 0$, $\nu_q(C_{y,q}) > 0$, and $sign(\Delta'(C_{y,q})) = sign(\nu'_q(C_{y,q})) = sign(C_{y,q})$, we have $F'(C_{y,q}) > 0$ for all $C_{y,q}$.

- $F''(C_{y,q}) > 0$ when $C_{y,q} > 0$ and $F''(C_{y,q}) < 0$ when $C_{y,q} < 0$. This follows from the fact that

$$F''(C_{y,q}) = 2[h_y g_y \Delta'(C_{y,q}) + h_q g_q \nu'_q(C_{y,q})] + [h_y g_y \Delta''(C_{y,q}) + h_q g_q \nu''_q(C_{y,q})] C_{y,q}.$$

- Together, these three previous properties imply that $F(C_{y,q})$ is a "cubic-shaped" function—i.e., $F(C_{y,q})$ is shaped like $A + BX^3$ for $A, B > 0$. To be clear, $F(C_{y,q})$ is not actually a cubic function of $C_{y,q}$. It simply behaves qualitatively like a cubic function of $C_{y,q}$.

Knowing that $F(C_{y,q})$ is "cubic-shaped" allows us to characterize the solutions to $C_{y,q} = F(C_{y,q})$. We have the following results:

- **Positive solutions:** Since (i) $F(0) > 0$ and (ii) $F'(C_{y,q}) > 0$ and $F''(C_{y,q}) > 0$ when $C_{y,q} > 0$, there are no positive solutions if $F'(0) > 1$. If $F'(0) < 1$, positive solutions may or may not exist. If there are positive solutions, there can be one or two solutions: a smaller stable solution (at which $F'(C_{y,q}) < 1$) and a larger unstable solution (at which $F'(C_{y,q}) > 1$). Since the existence of $\Delta(C_{y,q})$ and $\nu_q(C_{y,q})$ depends on $C_{y,q}$—they may exist for $(C_{y,q})^2$ small but not for $(C_{y,q})^2$ large—it is possible that no positive solutions exist. And, is possible that the smaller stable root exists, but that the larger unstable root does not exist.

- **Negative solutions:** Since (i) $F(0) > 0$ and (ii) $F'(C_{y,q}) > 0$ and $F''(C_{y,q}) < 0$ when $C_{y,q} > 0$, there is at most a single negative solution. Furthermore, if it exists, this negative solution must satisfy $F'(C^*_{y,q}) > 1$—i.e., it corresponds to an unstable solution to the fixed point problem. Since $F(C_{y,q})$ is cubic shaped, we have $C_{y,q} - F(C_{y,q}) < 0$ for $C_{y,q} \to -\infty$, assuming that $F(C_{y,q})$ continues to exist as $C_{y,q} \to -\infty$. However, since the existence of $\Delta(C_{y,q})$ and $\nu_q(C_{y,q})$ depends on $C_{y,q}$—they may exist for $(C_{y,q})^2$ small but not $(C_{y,q})^2$ large—it is possible that this negative unstable root does not exist.

- **Summary: Any stable solution is positive:** In summary, $C_{y,q} = F(C_{y,q})$ may have 0, 1, 2, or 3 solutions. If positive solutions exist, there is one stable positive solution (the smaller positive solution) and potentially one additional unstable positive solution (the larger positive solution). There is at most one negative solution to $C_{y,q} = F(C_{y,q})$ which is always unstable. *Thus, any stable solution is positive.* If a stable solution exists, we call this solution $\hat{C}_{y,q} > 0$.

- **Conditions to ensure existence of a positive stable solution:** A positive stable solution will never exist if $1 < F'(0)$. Therefore, a necessary—but not sufficient—condition for a positive stable solution to exist is that:

$$1 > F'(0) = g_y h_y \Delta(0, \tau) + g_q h_q \nu_q(0, \tau).$$

  Since both $\Delta(0, \tau)$ and $\nu_q(0, \tau)$ are decreasing in τ, this implies that we need τ sufficiently large. So we need $\tau > \hat{\tau}_{C_{y,q}}$ where $\hat{\tau}_{C_{y,q}}$ is implicitly defined as the solution to $1 = F'(0, \tau)$. However, knowing that $\tau > \hat{\tau}_{C_{y,q}}$ is necessary—but not sufficient—to ensure the existence of a positive stable solution. Furthermore, it is clear that only the positive stable solution exists for $\tau^{-1}\sigma_{s^y} \to 0$ and $\tau^{-1}\sigma_{s^q} \to 0$, since

$$\lim_{\tau^{-1}\sigma_{s^y}\to 0,\ \tau^{-1}\sigma_{s^q}\to 0} F\left(C_{y,q}\right) = \frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2\left(1-\rho\right) = \text{ Constant} > 0.$$

Step #4: If a positive stable solution $\hat{C}_{y,q}$ exists, use it to solve for V_y, V_q, $C_{y,y*}$. First, we solve condition (15a) for V_y. We know $\hat{C}_{y,q}$ and $\hat{\Delta} = \Delta(\hat{C}_{y,q})$. To determine V_y we use condition (15a) and solve the following quadratic equation for V_y

$$\begin{aligned} 0 &= g_y^2\left(V_y\right)^2 + g_y^2(V_y - \hat{\Delta})^2 - V_y + \left(\left(\frac{\delta}{1-\delta\phi_i}\right)^2\sigma_i^2 + g_q^2(\hat{C}_{y,q})^2\right) \\ &= 2g_y^2\left(V_y\right)^2 - (1+2g_y^2\hat{\Delta})V_y + \left(\left(\frac{\delta}{1-\delta\phi_i}\right)^2\sigma_i^2 + g_q^2\left(\hat{C}_{y,q}\right)^2 + g_y^2(\hat{\Delta})^2\right). \end{aligned}$$

A real solution only exists if $\tau > \hat{\tau}_y(\hat{C}_{y,q})$. As usual, if we want the smaller, stable solution of this quadratic. Call this solution $\hat{V}_y$.

Second, we can read off the solution to V_q: it is $\hat{V}_q = \nu_q(\hat{C}_{y,q}) > 0$.

Third, we can read off the solution to $C_{y,y*}$: it is $\hat{C}_{y,y*} = \hat{V}_y - \hat{\Delta} < \hat{V}_y$. Somewhat surprisingly, when $\sigma_{s^q}^2 > 0$, we can have $\hat{C}_{y,y*} < 0$ because foreign exchange supply shocks push domestic and long-term yields in opposite directions. For instance, if $\sigma_{s^q}^2 > 0$, $\sigma_{s^y}^2 = \sigma_{s^{y*}}^2 = 0$, and $\rho = 0$, condition (15c) becomes

$$C_{y,y*} = -g_q^2\left(C_{y,q}\right)^2 < 0,$$

so we must have $\hat{C}_{y,y*} < 0$. However, when $\rho > 0$, the two long-term yields tend to move in the same direction because domestic and foreign short rates are positively correlated. As a result, we have $\hat{C}_{y,y*} > 0$ unless foreign exchange supply shocks are large and ρ is near zero.

Summary of solution. When $\sigma_{s^q}^2 > 0$, $\sigma_{s^y}^2 > 0$, and $\rho \in (0,1)$, we have

$$\hat{V}_y > \left(\frac{\delta}{1-\delta\phi_i}\right)^2\sigma_i^2 > 0,\ \hat{V}_q > \left(\frac{1}{1-\phi_i}\right)^2 2\sigma_i^2\left(1-\rho\right) > 0, \text{ and } \hat{C}_{y,q} > \frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2\left(1-\rho\right) > 0$$

in the model's unique stable equilibrium. Naturally, once we add supply shocks, the equilibrium volatility of all three excess returns exceeds that in the absence of supply shocks. And, the equilibrium covariance between the excess returns on domestic bonds and the FX carry trade is positive—i.e., $\hat{C}_{y,q} = Cov_t\left[rx_{t+1}^y, rx_{t+1}^q\right] > 0$—and exceeds that in the absence of supply shocks. However, we can have $\hat{C}_{y,y*} < 0$ in the unique stable equilibrium. Specifically, if $\sigma_{s^q}^2 > 0$ and $\sigma_{s^y}^2 = \sigma_{s^{y*}}^2 = 0$, then we have $\hat{C}_{y,y*} < 0$ as $\rho \to 0$.

Solution properties. Consider a stable solution of the model. We show that $\hat{V}_q - 2\hat{C}_{y,q} > 0$ and $\hat{C}_{y,q} + \hat{C}_{y,y*} - \hat{V}_y > 0$ when $\delta < 1$ and $\rho < 1$. First, using the equations in (15), note that

$$
\begin{aligned}
& \hat{C}_{y,q} + \hat{C}_{y,y*} - \hat{V}_y \\
= \ & \frac{\delta\sigma_i^2 (1-\delta)(1-\rho)}{(1-\phi_i)(\delta\phi_i - 1)^2} + [(h_y g_y(\hat{V}_y - \hat{C}_{y,y*}) + h_q g_q \hat{V}_q)\hat{C}_{y,q} - 2g_q^2(\hat{C}_{y,q})^2 - g_y^2(\hat{V}_y - \hat{C}_{y,y*})^2] \\
> \ & \frac{\delta\sigma_i^2 (1-\delta)(1-\rho)}{(1-\phi_i)(1-\delta\phi_i)^2} + g_y^2(\hat{V}_y - \hat{C}_{y,y*}) \times [\hat{C}_{y,q} + \hat{C}_{y,y*} - \hat{V}_y] + g_q^2 \hat{C}_{y,q} \times [\hat{V}_q - 2\hat{C}_{y,q}],
\end{aligned}
$$

where the last line follows from the facts that the hs are larger than the gs, $\hat{V}_y - \hat{C}_{y,y*} > 0$, and $\hat{C}_{y,q} > 0$. Since $g_y^2(\hat{V}_y - \hat{C}_{y,y*}) < 1$ in equilibrium (see above), we have

$$
[\hat{C}_{y,q} + \hat{C}_{y,y*} - \hat{V}_y] > \overbrace{\frac{\delta\sigma_i^2 (1-\delta)(1-\rho)}{(1-\phi_i)(1-\delta\phi_i)^2} \frac{1}{1 - g_y^2(\hat{V}_y - \hat{C}_{y,y*})}}^{a>0} + \overbrace{\frac{g_q^2 \hat{C}_{y,q}}{1 - g_y^2(\hat{V}_y - \hat{C}_{y,y*})}}^{b>0} \times [\hat{V}_q - 2\hat{C}_{y,q}]. \quad (19)
$$

Proceeding similarly, we have

$$
\begin{aligned}
& [\hat{V}_q - 2\hat{C}_{y,q}] \\
= \ & \frac{2\sigma_i^2 (1-\delta)(1-\rho)}{(1-\phi_i)^2(1-\delta\phi_i)} + [2h_y^2(\hat{C}_{y,q})^2 + h_q^2(\hat{V}_q)^2 - 2(h_y g_y(\hat{V}_y - \hat{C}_{y,y*}) + h_q g_q \hat{V}_q)\hat{C}_{y,q}] \\
> \ & \frac{2\sigma_i^2 (1-\delta)(1-\rho)}{(1-\phi_i)^2(1-\delta\phi_i)} + 2h_y^2 \hat{C}_{y,q} \times [\hat{C}_{y,q} + \hat{C}_{y,y*} - \hat{V}_y] + h_q^2 \hat{V}_q \times [\hat{V}_q - 2\hat{C}_{y,q}].
\end{aligned}
$$

Since $h_q^2 \hat{V}_q < 1$ in equilibrium (see above), we have

$$
[\hat{V}_q - 2\hat{C}_{y,q}] > \overbrace{\frac{2\sigma_i^2 (1-\delta)(1-\rho)}{(1-\phi_i)^2(1-\delta\phi_i)} \frac{1}{1 - h_q^2 \hat{V}_q}}^{c>0} + \overbrace{\frac{2h_y^2 \hat{C}_{y,q}}{1 - h_q^2 \hat{V}_q}}^{d>0} \times [\hat{C}_{y,q} + \hat{C}_{y,y*} - \hat{V}_y]. \quad (20)
$$

Treating a, b, c, and d as fixed positive constants and $[\hat{C}_{y,q} + \hat{C}_{y,y*} - \hat{V}_y]$ and $[\hat{V}_q - 2\hat{C}_{y,q}]$ as unknowns to be characterized, we have shown that

$$
[\hat{V}_q - 2\hat{C}_{y,q}]/d - c/d > [\hat{C}_{y,q} + \hat{C}_{y,y*} - \hat{V}_y] > a + b \times [\hat{V}_q - 2\hat{C}_{y,q}]. \quad (21)
$$

Thus, it follows that the equilibrium solution must satisfy $[\hat{V}_q - 2\hat{C}_{y,q}] > 0$ and $[\hat{C}_{y,q} + \hat{C}_{y,y*} - \hat{V}_y] > 0$ when $\delta < 1$ and $\rho < 1$.

Additional results for Subsection 3.3.3 An analogous set of results to those presented in Section 3.3.3 also hold when $\sigma_{s^q}^2 > 0$. In this case, we have

$$
E_t\left[rx_{t+1}^q\right] = \overbrace{\left[\tau^{-1} C_{y,q}\right]}^{>0} \times (s_t^y - s_t^{y*}) + \overbrace{\left[\tau^{-1} V_q\right]}^{>0} \times s_t^q, \quad (22)
$$

$$
E_t\left[rx_{t+1}^y - rx_{t+1}^{y*}\right] = \overbrace{\left[\tau^{-1}\left(V_y - C_{y,y^*}\right)\right]}^{>0} \times (s_t^y - s_t^{y*}) + \overbrace{\left[2\tau^{-1} C_{y,q}\right]}^{>0} \times s_t^q. \quad (23)
$$

Thus, both (i) the expected return on FX carry trade and (ii) difference between domestic and foreign bond risk premia are increasing in (a) the difference in domestic and foreign bond supply and (b) the supply of FX carry trade. Combining these equations, we obtain

$$E_t\left[rx^q_{t+1}\right] = \overbrace{\left[\frac{C_{y,q}}{V_y - C_{y,y^*}}\right]}^{>0} \times E_t\left[rx^y_{t+1} - rx^{y*}_{t+1}\right] + \overbrace{\left[\tau^{-1}\left(\frac{V_q\left(V_y - C_{y,y^*}\right) - 2\left(C_{y,q}\right)^2}{V_y - C_{y,y^*}}\right)\right]}^{>0} \times s^q_t. \tag{24}$$

As in Section 3.3.3, the expected return on the FX carry trade is increasing in the difference between domestic and foreign risk premia. However, there is now a second term that reflects the impact of FX supply on the FX carry trade over and above the impact that FX supply has on the difference in bond risk premia.[8]

Finally, the expected return on the long-term carry trade is

$$E_t\left[rx^q_{t+1} + \left(rx^{y*}_{t+1} - rx^y_{t+1}\right)\right] = \overbrace{\left[\tau^{-1}\left(C_{y,q} - \left(V_y - C_{y,y^*}\right)\right)\right]}^{\in(0,\tau^{-1}C_{q,q})} \times \left(s^y_t - s^{y*}_t\right) + \overbrace{\left[\tau^{-1}\left(V_q - 2C_{y,q}\right)\right]}^{\in(0,\tau^{-1}V_q)} \times s^q_t. \tag{25}$$

Thus, comparing equations (22) and (25), we see that the expected return on the long-term FX carry trade is less responsive to movements in $(s^y_t - s^{y*}_t)$ and s^q_t—and, hence, less variable over time—than that on the short-term FX carry trade. Specifically, since $(V_y - C_{y,y^*}) > 0$ and $C_{y,q} > 0$, we have $C_{y,q} - (V_y - C_{y,y^*}) < C_{y,q}$ and $V_q - 2C_{y,q} < V_q$. And, as shown above, both terms are still positive in the presence of supply risk.[9]

Finally, $\lim_{\delta\to1}\left[C_{y,q} + C_{y,y^*} - V_y\right] = \lim_{\rho\to1}\left[C_{y,q} + C_{y,y^*} - V_y\right] = 0$ and $\lim_{\delta\to1}\left[V_q - 2C_{y,q}\right] = \lim_{\rho\to1}\left[V_q - 2C\right.$ 0. Since $Var_t\left[rx^q_{t+1} + \left(rx^{y*}_{t+1} - rx^y_{t+1}\right)\right] = V_q + 2V_y - 2C_{y,y*} - 4C_{y,q}$, this implies that $\lim_{\delta\to1} Var_t\left[rx^q_{t+1} + \left(rx^{y*}_{t+1} - \right.\right.$ $\lim_{\rho\to1} Var_t\left[rx^q_{t+1} + \left(rx^{y*}_{t+1} - rx^y_{t+1}\right)\right] = 0$. In the limiting cases where long-term bonds have infinite duration or whether the domestic and foreign short rates are perfectly correlated, the long-term FX carry trade is completely riskless and therefore earns a zero excess return.

Of course, this result assumes that there are no independent shocks to long-run FX fundamentals $(\sigma^2_{q\infty} = 0)$ as in our baseline model. When there are independent shocks to long-run FX fundamentals $(\sigma^2_{q\infty} > 0)$, we still have $\lim_{\delta\to1}\left[C_{y,q} + C_{y,y^*} - V_y\right] = \lim_{\rho\to1}\left[C_{y,q} + C_{y,y^*} - V_y\right] = 0$. However, when $\sigma^2_{q\infty} > 0$, we now have $\lim_{\delta\to1}\left[V_q - 2C_{y,q}\right] = \lim_{\rho\to1}\left[V_q - 2C_{y,q}\right] > 0$ and $\lim_{\delta\to1} Var_t\left[rx^q_{t+1} + \left(rx^{y*}_{t+1} - rx^y_{t+1}\right)\right] = \lim_{\rho\to1} Var_t\left[rx^q_{t+1} + \left(rx^{y*}_{t+1} - rx^y_{t+1}\right)\right] > 0$.

[8] In the presence of supply risk, we have $\det(\mathbf{V}) = (V_y + C_{y,y^*})\left[V_q\left(V_y - C_{y,y^*}\right) - 2\left(C_{y,q}\right)^2\right] > 0$ Thus, since $(V_y + C_{y,y^*}) > 0$, we have $V_q\left(V_y - C_{y,y^*}\right) - 2\left(C_{y,q}\right)^2 > 0$ and the term multiplying s^q_t is positive.

[9] To interpret this expression, note that $\left[C_{y,q} - \left(V_y - C_{y,y^*}\right)\right] = Cov_t\left[rx^q_{t+1} + \left(rx^{y*}_{t+1} - rx^y_{t+1}\right), rx^y_{t+1}\right]$ and $\left[V_q - 2C_{y,q}\right] = Cov_t\left[rx^q_{t+1} + \left(rx^{y*}_{t+1} - rx^y_{t+1}\right), rx^q_{t+1}\right]$.

A.5.3 Solution with only bond supply shocks

When $\sigma_{s_q} = 0$, we have $g_q = h_q = 0$ and the system of equations simplifies further. Specifically, the fixed point problem reduces to the following system of four equations in four unknowns:

$$
\begin{aligned}
V_y &= \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \sigma_i^2 + \left[g_y^2 (V_y)^2 + g_y^2 (C_{y,y^*})^2\right] && \text{(26a)}\\
V_q &= \left(\frac{1}{1-\phi_i}\right)^2 2\sigma_i^2 (1-\rho) + \left[2h_y^2 (C_{y,q})^2\right] && \text{(26b)}\\
C_{y,y*} &= \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \rho\sigma_i^2 + \left[2g_y^2 V_y C_{y,y^*}\right] && \text{(26c)}\\
C_{y,q} &= \frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2 (1-\rho) + \left[h_y g_y (V_y - C_{y,y^*})\right] C_{y,q} && \text{(26d)}
\end{aligned}
$$

We now assume that $\rho \in (0,1)$. This system can be solved using the following sequence of steps.

Step #1: Solve for $\Delta \equiv V_y - C_{y,y^*}$. Subtracting the condition (26c) for C_{y,y^*} the from condition (26a) for V_y, we obtain

$$V_y - C_{y,y*} = \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \sigma_i^2 (1-\rho) + g_y^2 (V_y - C_{y,y*})^2 \geq 0.$$

We can solve the resulting quadratic for $\Delta \equiv V_y - C_{y,y^*} > 0$. The quadratic is

$$0 = g_y^2 \Delta^2 - \Delta + \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \sigma_i^2 (1-\rho), \tag{27}$$

which has a real solution if and only if

$$\frac{\tau}{2} > \sqrt{1-\rho} \times \frac{\delta}{1-\delta\phi_i}\sigma_i \times \frac{\delta}{1-\delta\phi_{s^y}}\sigma_{s^y}. \tag{28}$$

The model's stable equilibrium corresponds to the smaller root of this quadratic and is given by

$$\hat{\Delta} = \frac{1 - \sqrt{1 - 4g_y^2 \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \sigma_i^2 (1-\rho)}}{2g_y^2} > 0. \tag{29}$$

This stable solution $\hat{\Delta}$ converges to $\hat{\Delta} = 0$ as $\rho \to 1$ and to $\hat{\Delta} = (\delta\sigma_i / (1-\delta\phi_i))^2 (1-\rho)$ as $\tau^{-1}\sigma_{s^y} \to 0$. This stable solution also has natural comparative statics: $\partial\hat{\Delta}/\partial\tau < 0$, $\partial\hat{\Delta}/\partial\sigma_{s^y} > 0$, $\partial\hat{\Delta}/\partial\phi_{s^y} > 0$, $\partial\hat{\Delta}/\partial\sigma_i > 0$, $\partial\hat{\Delta}/\partial\phi_i > 0$, and $\partial\hat{\Delta}/\partial\rho < 0$.

Step #2: Substitute $\hat{\Delta}$ into condition (26d) for $C_{y,q}$ to obtain a solution for $C_{y,q}$. We substitute the solution for Δ into condition (26d) for $C_{y,q}$ and solve to obtain a solution for $C_{y,q}$. Doing so we obtain the following solution for $C_{y,q}$:

$$\hat{C}_{y,q} = \frac{\left(\frac{\delta\sigma_i}{1-\delta\phi_i}\right)\left(\frac{\sigma_i}{1-\phi_i}\right)(1-\rho)}{1-g_y h_y \hat{\Delta}} = \frac{\left(\frac{\delta\sigma_i}{1-\delta\phi_i}\right)\left(\frac{\sigma_i}{1-\phi_i}\right)(1-\rho)}{1-\frac{1}{2}\frac{\frac{1}{1-\phi_{s^y}}}{\frac{\delta}{1-\delta\phi_{s^y}}}\left(1-\sqrt{1-4\left(\frac{\delta\tau^{-1}\sigma_{s^y}}{1-\delta\phi_{s^y}}\right)^2\left(\frac{\delta\sigma_i}{1-\delta\phi_i}\right)^2(1-\rho)}\right)}. \tag{30}$$

We can show that we must have

$$1 > g_y h_y \hat{\Delta} = \left(\frac{\tau^{-1}\sigma_{s^y}}{1-\phi_{s^y}}\right)\left(\frac{\delta\tau^{-1}\sigma_{s^y}}{1-\delta\phi_{s^y}}\right)(\hat{V}_y - \hat{C}_{y,y^*}),$$

in any stable equilibrium. Thus, we have $\hat{C}_{y,q} \geq 0$ in any stable equilibrium and $\hat{C}_{y,q} > 0$ when $\rho \in [0,1)$. Intuitively, if this condition doesn't hold then, from condition (26d), a small perturbation to the equilibrium value of $C_{y,q}$ leads to larger and larger changes in $C_{y,q}$, indicating that the equilibrium solution is unstable. Formally, we can show that $g_y h_y \hat{\Delta} \geq 0$ is one of the eigenvalues of the Jacobian matrix of the relevant fixed point problem, so we must have $g_y h_y \hat{\Delta} < 1$ in any stable equilibrium.

Finally, it is easy to see that $\partial \hat{C}_{y,q}/\partial\rho < 0$ and that $\hat{C}_{y,q} = 0$ when $\rho = 1$.

Step #3: Substitute $\hat{\Delta}$ into condition (26c) for C_{y,y^*} to obtain a solution for C_{y,y^*}. Proceeding similarly, we substitute $V_y = \hat{\Delta} + C_{y,y^*}$ into condition (26c), we obtain

$$C_{y,y*} = \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \rho\sigma_i^2 + 2g_y^2(\hat{\Delta} + C_{y,y^*})C_{y,y^*}.$$

Thus, we need to solve the following quadratic in C_{y,y^*}:

$$0 = 2g_y^2\left(C_{y,y^*}\right)^2 + (2g_y^2\hat{\Delta} - 1)C_{y,y^*} + \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \rho\sigma_i^2. \tag{31}$$

We can show that $2g_y^2\hat{\Delta} = 2g_y^2\left(V_y - C_{y,y^*}\right) < 1$ in any stable equilibrium. Specifically, we can show that $2g_y^2\Delta = 2g_y^2\left(V_y - C_{y,y^*}\right) \geq 0$ is one of the eigenvalues of the Jacobian matrix of the relevant fixed point operator, so we must have $2g_y^2\Delta = 2g_y^2\left(V_y - C_{y,y^*}\right) < 1$ in any stable equilibrium. Thus, so long as $\rho > 0$, it follows that $\hat{C}_{y,y^*} > 0$ in any stable equilibrium.

Since $1 - 2g_y^2\hat{\Delta} > 0$, a real solution exists so long as $1 - 2g_y^2\hat{\Delta} > 2\sqrt{2}g_y\left(\delta/\left(1-\delta\phi_i\right)\right)\sigma_i\sqrt{\rho}$. Using the expressions for $\hat{\Delta}$ and g_y, this is equivalent to

$$\frac{\tau}{2} > \sqrt{1+\rho} \times \frac{\delta}{1-\delta\phi_i}\sigma_i \times \frac{\delta}{1-\delta\phi_{s^y}}\sigma_{s^y}.$$

The relevant stable solution for C_{y,y^*} is

$$\hat{C}_{y,y^*} = \frac{(1-2g_y^2\hat{\Delta}) - \sqrt{(1-2g_y^2\hat{\Delta})^2 - 8g_y^2\left(\frac{\delta}{1-\delta\phi_i}\right)^2\rho\sigma_i^2}}{4g_y^2.} > 0.$$

At this stable solution, we have $\hat{C}_{y,y^*} \to 0$ when $\rho \to 0$ and $\hat{C}_{y,y^*} \to \hat{V}_y$ when $\rho \to 1$.

Step #4: Obtain solutions for V_y and V_q. The solution for V_y is trivially given by

$$\hat{V}_y = \hat{\Delta} + \hat{C}_{y,y^*} > 0.$$

And, the solution for V_q is given by

$$\hat{V}_q = \left(\frac{1}{1-\phi_i}\right)^2 2\sigma_i^2 (1-\rho) + 2\left(\frac{\tau^{-1}\sigma_{s^y}}{1-\phi_{s^y}}\right)^2 (\hat{C}_{y,q})^2 > 0.$$

Solution summary. When $\sigma_{s^q}^2 = 0$, $\sigma_{s^y}^2 > 0$, and $\rho \in (0,1)$, we have $\hat{V}_y > 0$, $\hat{V}_q > 0$, $\hat{C}_{y,q} > 0$, and $\hat{C}_{y,y^*} > (\delta/(1-\delta\phi_i))^2 \rho\sigma_i^2 > 0$ in the model's unique stable equilibrium. Thus, in this case, all equilibrium variance and covariances exceed those in the absence of supply risk.

Solution properties. We are interested in the term in square brackets in

$$E_t\left[rx_{t+1}^q\right] = \left[\frac{\hat{C}_{y,q}}{\hat{V}_y - \hat{C}_{y,y^*}}\right] \times E_t\left[rx_{t+1}^y - rx_{t+1}^{y*}\right]. \tag{32}$$

Since $\hat{C}_{y,q} > 0$ and $\hat{V}_y - \hat{C}_{y,y^*} > 0$ in any stable equilibrium, this quantity is obviously positive. We now show that $[\hat{C}_{y,q}/(\hat{V}_y - \hat{C}_{y,y^*})] > 1$. Using equation (27), we can rewrite this term as

$$\frac{\hat{C}_{y,q}}{\hat{V}_y - \hat{C}_{y,y^*}} = \frac{1}{\hat{\Delta}} \frac{\left(\frac{\delta}{1-\delta\phi_i}\sigma_i\right)\left(\frac{1}{1-\phi_i}\sigma_i\right)(1-\rho)}{1-\left(\frac{\tau^{-1}\sigma_{sy}}{1-\phi_{sy}}\right)\left(\frac{\delta\tau^{-1}\sigma_{sy}}{1-\delta\phi_{sy}}\right)\hat{\Delta}} = \frac{\left(\frac{\delta}{1-\delta\phi_i}\sigma_i\right)\left(\frac{1}{1-\phi_i}\sigma_i\right)}{\left(\frac{\delta}{1-\delta\phi_i}\sigma_i\right)^2 - \frac{1-\delta}{\delta(1-\phi_{sy})}\left(\hat{\Delta}^\rho - \left(\frac{\delta}{1-\delta\phi_i}\right)^2\sigma_i^2\right)}.$$

where $\hat{\Delta}^\rho \equiv \hat{\Delta}/(1-\rho)$ is the smaller root of the following quadratic equation

$$0 = \left(\frac{\delta\tau^{-1}\sigma_{s^y}}{1-\delta\phi_{s^y}}\right)^2 (1-\rho)(\Delta^\rho)^2 - \Delta^\rho + \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \sigma_i^2.$$

Since $\hat{\Delta}^\rho > (\delta/(1-\delta\phi_i))^2 \sigma_i^2$ when $\rho < 1$, it follows that

$$\frac{\hat{C}_{y,q}}{\hat{V}_y - \hat{C}_{y,y^*}} = \frac{\left(\frac{\delta}{1-\delta\phi_i}\sigma_i\right)\left(\frac{1}{1-\phi_i}\sigma_i\right)}{\left(\frac{\delta}{1-\delta\phi_i}\sigma_i\right)^2 - \frac{1-\delta}{\delta(1-\phi_{sy})}\left(\hat{\Delta}^\rho - \left(\frac{\delta}{1-\delta\phi_i}\right)^2\sigma_i^2\right)} > \frac{\left(\frac{\delta}{1-\delta\phi_i}\sigma_i\right)\left(\frac{1}{1-\phi_i}\sigma_i\right)}{\left(\frac{\delta}{1-\delta\phi_i}\sigma_i\right)^2} = \frac{\frac{1}{1-\phi_i}}{\frac{\delta}{1-\delta\phi_i}} > 1.$$

It is also easy to see that $\partial\hat{\Delta}^\rho/\partial\rho < 0$ when $\sigma_{s^y}^2 > 0$, thus we have

$$\frac{\partial}{\partial\rho}\left[\frac{\hat{C}_{y,q}}{\hat{V}_y - \hat{C}_{y,y^*}}\right] \propto \frac{\partial\hat{\Delta}^\rho}{\partial\rho} < 0$$

when $\sigma_{s^y}^2 > 0$. Finally, since $\hat{C}_{y,q} > (\hat{V}_y - \hat{C}_{y,y^*})$, it follows that

$$E_t\left[rx_{t+1}^q + \left(rx_{t+1}^{y*} - rx_{t+1}^y\right)\right] = \overbrace{[\tau^{-1}(\hat{C}_{y,q} - (\hat{V}_y - \hat{C}_{y,y^*}))]}^{>0} \times \left(s_t^y - s_t^{y*}\right) = \overbrace{\left[1 - \frac{\hat{V}_y - \hat{C}_{y,y^*}}{\hat{C}_{y,q}}\right]}^{\in(0,1)} \times E_t\left[rx_{t+1}^q\right].$$

A.5.4 Allowing for asymmetries between the two countries

This subsection discusses how the results of our baseline model in Section 3 generalize if we allow the two countries to have different short rate and bond supply processes.

First, since the stable equilibrium is continuous in the model's underlying parameters, Proposition 2 implies that $C_{y,q} > 0$ whenever $\rho < 1$ and the short rates and bond supply follow sufficiently symmetric processes. For example, while $Cov_t[rx^{y}_{t+1}, rx^{q}_{t+1}] \neq -Cov_t[rx^{y*}_{t+1}, rx^{q}_{t+1}]$, we still have $Cov_t[rx^{y}_{t+1}, rx^{q}_{t+1}] > 0$ and $Cov_t[rx^{y*}_{t+1}, rx^{q}_{t+1}] < 0$ if there are moderate asymmetries between the domestic and foreign short rate and bond supply processes—e.g., there can be moderate differences in either the volatilities or persistences. Furthermore, $C_{y,y^*} > 0$ whenever $\rho > 0$, the short rates and bond supply follow sufficiently symmetric processes, and when FX supply risk is sufficiently small relative to ρ.

However, things grow more complicated if we allow for highly asymmetric short rate and bond supply processes. For instance, with highly asymmetric short rate processes, the sign of $C_{y,q}$ is ambiguous and the sign of $C_{y*,q}$ need not be opposite that of $C_{y,q}$. For instance, suppose that $\sigma^2_{i^*} \equiv Var_t[\varepsilon_{i^*_{t+1}}] \neq Var_t[\varepsilon_{i_{t+1}}] \equiv \sigma^2_i$, but the two short rates share the same persistence ϕ_i. Then, focusing on the limit where there is no supply risk for simplicity, we have

$$C_{y,y^*} = \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \rho\sigma_i\sigma_{i^*}, \tag{33a}$$

$$C_{y,q} = \frac{1}{1-\phi_i}\frac{\delta}{1-\delta\phi_i}\sigma^2_i\left(1-\rho\frac{\sigma_{i^*}}{\sigma_i}\right) \tag{33b}$$

$$C_{y^*,q} = -\frac{1}{1-\phi_i}\frac{\delta}{1-\delta\phi_i}\sigma^2_{i^*}\left(1-\rho\frac{\sigma_i}{\sigma_{i^*}}\right). \tag{33c}$$

While we still have $C_{y,y^*} > 0$ so long as $\rho > 0$, the behavior of $C_{y,q}$ and $C_{y^*,q}$ is more complicated. Noting that $\rho\sigma_{i^*}/\sigma_i$ ($\rho\sigma_i/\sigma_{i^*}$) is the coefficient from a regression of i^*_t on i_t (i_t on i^*_t), there are now three possible case:[10]

1. If $1 > \max\{\rho\sigma_{i^*}/\sigma_i, \rho\sigma_i/\sigma_{i^*}\}$—i.e., if the short rates are sufficiently symmetric, $C_{y,q} > 0$ and $C_{y^*,q} < 0$.
2. If $\rho\sigma_{i^*}/\sigma_i > 1 > \rho\sigma_i/\sigma_{i^*}$—i.e., if foreign short rates move more than one-for-one with domestic short rates, then $C_{y,q} < 0$ and $C_{y^*,q} < 0$.
3. If $\rho\sigma_i/\sigma_{i^*} > 1 > \rho\sigma_{i^*}/\sigma_i$—i.e., if domestic short rates move more than one-for-one with foreign short rates, $C_{y,q} > 0$ and $C_{y^*,q} > 0$.

Thus, in the event of a positive shock to the supply of long-term dollar bonds, foreign currencies with $\rho\sigma_{i^*}/\sigma_i < 1$ would be expected to *depreciate* against the dollar on impact and then appreciate going forward: this is the case emphasized in the main text. By contrast, foreign currencies with $\rho\sigma_{i^*}/\sigma_i > 1$ would be expected to *appreciate* versus the dollar on impact and then depreciate going forward. To see the intuition, suppose that $\rho\sigma_{i^*}/\sigma_i > 1 > \rho\sigma_i/\sigma_{i^*}$, so foreign short rates move more than one-for-one with domestic short rates. Here an increase in the supply of long-term domestic bonds leads to a larger increase in the price of foreign short rate risk than in the price of domestic foreign short rate risk. Since foreign exchange has a positive exposure to domestic short rates and a negative—and opposite—exposure to foreign short rates, the increase in domestic bond supply actually reduces the expected future return on foreign exchange, leading foreign currency to appreciate today. And, since an increase in foreign bond supply also has a larger impact on the price of foreign short rate risk, such a shock also leads foreign currency to appreciate.

[10]However, since $\min\{\rho\sigma_{i^*}/\sigma_i, \rho\sigma_i/\sigma_{i^*}\} < 1$, we can never have $C_{y,q} < 0$ and $C_{y^*,q} > 0$.

A.6 A unified approach to carry trade returns

In Section 3.4, we extend the baseline model to explain the linkages between expected carry trade returns and short-term interest rates. To do so, we assume that the total net supplies that must be absorbed by arbitrageurs are

$$\mathbf{n}_t = \mathbf{s}_t + \mathbf{S}_2\mathbf{y}_t \text{ where } \mathbf{S}_2 \equiv \begin{bmatrix} -S_y & 0 & 0 \\ 0 & -S_y & 0 \\ 0 & 0 & S_q \end{bmatrix}. \tag{34}$$

Thus, following Vayanos and Vila (2019), we assume the supplies of long-term bonds in both currencies are decreasing in the relevant long-term bond yield; and, following Gabaix and Maggiori (2015), we assume the supply of the FX carry trade is increasing in strength of the foreign currency. Since $\mathbf{s}_t = \mathbf{s}_0 + \mathbf{S}_1\mathbf{z}_t$ and $\mathbf{y}_t = \mathbf{a} + \mathbf{A}\mathbf{z}_t$, we have $\mathbf{n}_t = (\mathbf{s}_0 + \mathbf{S}_2\mathbf{a}) + (\mathbf{S_1} + \mathbf{S_2}\mathbf{A})\,\mathbf{z}_t$.

The market clearing condition for the extended model, namely $E_t\left[\mathbf{rx}_{t+1}\right] = \tau^{-1}\mathbf{V}\mathbf{n}_t$, can be written as

$$[\mathbf{B}_0\mathbf{a} + \mathbf{B}_1\mathbf{a} + \mathbf{r}_0] + [\mathbf{B}_0\mathbf{A} + \mathbf{B}_1\mathbf{A}\boldsymbol{\Phi} + \mathbf{R}_1]\,\mathbf{z}_t = \tau^{-1}\mathbf{V}\left(\mathbf{s}_0 + \mathbf{S}_2\mathbf{a}\right) + \tau^{-1}\mathbf{V}\left(\mathbf{S_1} + \mathbf{S_2}\mathbf{A}\right)\mathbf{z}_t, \tag{35}$$

where $\mathbf{V} = (\mathbf{B}_1\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}'\mathbf{B}_1')$.

Matching slope coefficients in equation (35), we have

$$[\mathbf{B}_0\mathbf{E} + \mathbf{B}_1\mathbf{E}\boldsymbol{\Phi}] \circ \mathbf{A} - \tau^{-1}\mathbf{V}\mathbf{S}_2\mathbf{A} = \tau^{-1}\mathbf{V}\mathbf{S_1} - \mathbf{R}_1.$$

To solve for $\mathbf{A}$, we vectorize this condition to obtain

$$diag\left(vec\left(\mathbf{B}_0\mathbf{E} + \mathbf{B}_1\mathbf{E}\boldsymbol{\Phi}\right)\right) vec\left(\mathbf{A}\right) - \tau^{-1}\left(\mathbf{I}_5 \otimes \left(\mathbf{V}\mathbf{S}_2\right)\right) vec\left(\mathbf{A}\right) = vec\left(\tau^{-1}\mathbf{V}\mathbf{S_1} - \mathbf{R}_1\right)$$

where $\mathbf{I}_5$ is the 5×5 identity matrix and $\otimes$ denotes a Kronecker product. Solving this equation $vec\left(\mathbf{A}\right)$, we require

$$vec\left(\mathbf{A}\right) = \left[diag\left(vec\left(\mathbf{B}_0\mathbf{E} + \mathbf{B}_1\mathbf{E}\boldsymbol{\Phi}\right)\right) - \tau^{-1}\left(\mathbf{I}_5 \otimes \left(\mathbf{V}\mathbf{S}_2\right)\right)\right]^{-1} vec\left(\tau^{-1}\mathbf{V}\mathbf{S_1} - \mathbf{R}_1\right), \tag{36}$$

where we note that the matrix is square brackets is block-diagonal. Finally, since $\mathbf{V} = \mathbf{B}_1\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}'\mathbf{B}_1'$, we obtain the following fixed-point point problem in $\mathbf{A}$:

$$vec\left(\mathbf{A}\right) = \left[diag\left(vec\left(\mathbf{B}_0\mathbf{E} + \mathbf{B}_1\mathbf{E}\boldsymbol{\Phi}\right)\right) - \tau^{-1}\left(\mathbf{I}_5 \otimes \left(\mathbf{B}_1\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}'\mathbf{B}_1'\mathbf{S}_2\right)\right)\right]^{-1} vec\left(\tau^{-1}\mathbf{B}_1\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}'\mathbf{B}_1'\mathbf{S_1} - \mathbf{R}_1\right). \tag{37}$$

Natural, the fixed point problem in equation (36) reduces to that in equation (5) when $\mathbf{S}_2 = \mathbf{0}$.

Matching constant terms in equation (35), we obtain

$$\left(\mathbf{B}_0 + \mathbf{B}_1 - \tau^{-1}\mathbf{V}\mathbf{S}_2\right)\mathbf{a} = \left(\tau^{-1}\mathbf{V}\mathbf{s}_0 - \mathbf{r}_0\right).$$

This condition always allows us to pin down the steady state levels of bond yields, α_0^y and α_0^{y*}. When $S_q = 0$, $(\mathbf{B}_0 + \mathbf{B}_1 - \tau^{-1}\mathbf{V}\mathbf{S}_2)$ is singular and the steady-state level of exchange rates α_0^q is not pinned down as in the baseline model. However, when $S_q > 0$, $(\mathbf{B}_0 + \mathbf{B}_1 - \tau^{-1}\mathbf{V}\mathbf{S}_2)$ is invertible and α_0^q is pinned down. Specifically, since we have assumed the home and foreign countries are perfectly symmetric, we have $\alpha_0^q = 0$. (We would not have $\alpha_0^q = 0$ if the countries were not symmetric.) Intuitively, α_0^q is pinned down because the steady-state supply of the FX carry trade depends on the steady-state level of the exchange rate.

We are mainly interested in the loadings on i_t and i_t^*. Using equation (36), the loadings on i_t take the form

$$\begin{bmatrix} \alpha_i^y \\ \alpha_i^{y*} \\ \alpha_i^q \end{bmatrix} = \begin{bmatrix} \frac{1-\delta\phi_i}{1-\delta} + \tau^{-1}S_y V_y & \tau^{-1}S_y C_{y,y^*} & -\tau^{-1}S_q C_{y,q} \\ \tau^{-1}S_y C_{y,y^*} & \frac{1-\delta\phi_i}{1-\delta} + \tau^{-1}S_y V_y & \tau^{-1}S_q C_{y,q} \\ \tau^{-1}S_y C_{y,q} & -\tau^{-1}S_y C_{y,q} & -(1-\phi_i) - \tau^{-1}S_q V_q \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \tag{38}$$

and the loadings on i_t^* take the form

$$\begin{bmatrix} \alpha_{i^*}^y \\ \alpha_{i^*}^{y*} \\ \alpha_{i^*}^q \end{bmatrix} = \begin{bmatrix} \frac{1-\delta\phi_i}{1-\delta} + \tau^{-1}S_y V_y & \tau^{-1}S_y C_{y,y^*} & -\tau^{-1}S_q C_{y,q} \\ \tau^{-1}S_y C_{y,y^*} & \frac{1-\delta\phi_i}{1-\delta} + \tau^{-1}S_y V_y & \tau^{-1}S_q C_{y,q} \\ \tau^{-1}S_y C_{y,q} & -\tau^{-1}S_y C_{y,q} & -(1-\phi_i) - \tau^{-1}S_q V_q \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}. \tag{39}$$

Thus, it follows that

$$\alpha_{i^*}^{y*} = \alpha_i^y,\ \alpha_{i^*}^y = \alpha_i^{y*},\text{ and } \alpha_{i^*}^q = -\alpha_i^q. \tag{40}$$

We study the extended model in two specific cases:

- **Case #1:** $S_q > 0$ and $S_y = 0$.
- **Case #2:** $S_q = 0$ and $S_y > 0$.

A.6.1 Calculations for Case #1: $S_q > 0$ and $S_y = 0$.

General analysis When $S_q > 0$ and $S_y = 0$, equation (38) implies that the loadings on i_t are

$$\begin{bmatrix} \alpha_i^y \\ \alpha_i^{y*} \\ \alpha_i^q \end{bmatrix} = \begin{bmatrix} \frac{1-\delta}{1-\delta\phi_i} \frac{1-\phi_i+\tau^{-1}S_q V_q - \tau^{-1}S_q C_{y,q}}{1-\phi_i+\tau^{-1}S_q V_q} \\ \frac{1-\delta}{1-\delta\phi_i} \frac{\tau^{-1}S_q C_{y,q}}{1-\phi_i+\tau^{-1}S_q V_q} \\ -\frac{1}{1-\phi_i+\tau^{-1}S_q V_q} \end{bmatrix} \tag{41}$$

and the loading on i_t^* again satisfy equation (40). Using equations (41) and (40), we note that

$$\alpha_i^y + \alpha_{i^*}^y = \frac{1-\delta}{1-\delta\phi_i} < 1.$$

Bounding the quantities in equation (41), we see that:

1. $\alpha_i^y \in \left(0, \frac{1-\delta}{1-\delta\phi_i}\right)$ so long as (i) $0 < C_{y,q}$ and (ii) $\tau^{-1}S_q C_{y,q} < 1 - \phi_i + \tau^{-1}S_q V_q$;
   - $\alpha_i^y < \frac{1-\delta}{1-\delta\phi_i}$ so long as (i) $0 < C_{y,q}$;
   - $\alpha_i^y > 0$ so long as (ii) $\tau^{-1}S_q C_{y,q} < 1 - \phi_i + \tau^{-1}S_q V_q$;
2. $\alpha_i^{y*} > 0$ so long as $0 < C_{y,q}$;
3. $\alpha_i^q \in \left(-\frac{1}{1-\phi_i}, 0\right)$ since $V_q > 0$.

We now explore how changes in short-term interest rates impact equilibrium expected excess returns. Since $V_q > 0$, we have

$$\gamma_{i^*}^q \equiv \partial E_t\left[rx_{t+1}^q\right]/\partial i_t^* = (\phi_i - 1)\,\alpha_{i^*}^q + 1 = \frac{\tau^{-1}S_q V_q}{1-\phi_i+\tau^{-1}S_q V_q} > 0.$$

Symmetrically, we have $\gamma_i^q \equiv \partial E_t\left[rx_{t+1}^q\right]/\partial i_t = -\gamma_{i^*}^q$. Furthermore, so long as $C_{y,q} > 0$, we have

$$\begin{aligned}\gamma_i^y &\equiv \partial E_t\left[rx_{t+1}^y\right]/\partial i_t = \left(\frac{1-\delta\phi_i}{1-\delta}\right)\alpha_i^y - 1 = -\frac{\tau^{-1}S_qC_{y,q}}{1-\phi_i+\tau^{-1}S_qV_q} < 0,\\ \gamma_{i^*}^y &\equiv \partial E_t\left[rx_{t+1}^y\right]/\partial i_t^* = \left(\frac{1-\delta\phi_i}{1-\delta}\right)\alpha_{i^*}^y = \frac{\tau^{-1}S_qC_{y,q}}{1-\phi_i+\tau^{-1}S_qV_q} > 0.\end{aligned}$$

Symmetrically, we have $\gamma_{i*}^{y*} \equiv \partial E_t\left[rx_{t+1}^{y*}\right]/\partial i_t^* = \gamma_i^y$ and $\gamma_i^{y*} \equiv \partial E_t\left[rx_{t+1}^{y*}\right]/\partial i_t = \gamma_{i^*}^y$.

Closing the model in Case #1 when there are no independent supply shocks It is straightforward to analytically solve for the equilibrium variance and covariance terms in case where there are only shocks to the two short rates—-i.e., when $\sigma_{s^y}^2 = \sigma_{s^q}^2 = 0$. In this case, we can show that (i) $0 < C_{y,q}$ and (ii) $\tau^{-1}S_qC_{y,q} < 1-\phi_i+\tau^{-1}S_qV_q$, confirming that the bounds discussed above indeed hold.

Assume there are only short rate shocks. Making use of the facts that $\alpha_{i^*}^{y*} = \alpha_i^y$, $\alpha_{i^*}^y = \alpha_i^{y*}$, and $\alpha_{i^*}^q = -\alpha_i^q$, we have

$$\begin{aligned}rx_{t+1}^y - E_t\left[rx_{t+1}^y\right] &= -\frac{\delta}{1-\delta}\left(\alpha_i^y\varepsilon_{i,t+1} + \alpha_{i^*}^y\varepsilon_{i^*,t+1}\right)\\ rx_{t+1}^{y*} - E_t\left[rx_{t+1}^{y*}\right] &= -\frac{\delta}{1-\delta}\left(\alpha_{i^*}^y\varepsilon_{i,t+1} + \alpha_i^y\varepsilon_{i^*,t+1}\right)\\ rx_{t+1}^q - E_t\left[rx_{t+1}^q\right] &= \alpha_{i^*}^q\left(\varepsilon_{i^*,t+1} - \varepsilon_{i,t+1}\right).\end{aligned}$$

It is then easy to solve for the equilibrium level of V_q. We have

$$V_q = Var\left[\alpha_{i^*}^q\left(\varepsilon_{i^*,t+1} - \varepsilon_{i,t+1}\right)\right] = \left(\alpha_{i^*}^q\right)^2 2\sigma_i^2\left(1-\rho\right).$$

Using the fact that $\alpha_{i^*}^q = 1/\left(1-\phi_i+\tau^{-1}S_qV_q\right)$, we obtain the following fixed-point condition for V_q:

$$V_q = f\left(V_q\right) = \frac{2\sigma_i^2\left(1-\rho\right)}{\left(1-\phi_i+\tau^{-1}S_qV_q\right)^2} > 0.$$

Since $f\left(V_q\right) > 0$, $f'\left(V_q\right) < 0$, and $\lim_{V_q\to\infty} f\left(V_q\right) = 0$, it follows that there is a unique solution $\hat{V}_q > 0$. It is also easy to see that $\partial\hat{V}_q/\partial S_q < 0$, so we have $\hat{V}_q < 2\sigma_i^2\left(1-\rho\right)/\left(1-\phi_i\right)^2$. However, we have $\partial(S_q\hat{V}_q)/\partial S_q > 0$.

We now solve for $C_{y,q}$. We have

$$\begin{aligned}C_{y,q} &= -\frac{\delta}{1-\delta}Cov\left[\alpha_i^y\varepsilon_{i,t+1} + \alpha_{i^*}^y\varepsilon_{i^*,t+1}, \alpha_{i^*}^q\left(\varepsilon_{i^*,t+1} - \varepsilon_{i,t+1}\right)\right]\\ &= \frac{\delta}{1-\delta}\left(1-\rho\right)\sigma_i^2\alpha_{i^*}^q\left(\alpha_i^y - \alpha_{i^*}^y\right)\\ &= \frac{\left(1-\rho\right)\sigma_i^2}{1-\phi_i+\tau^{-1}S_qV_q}\frac{\delta}{1-\delta\phi_i}\left(\frac{1-\phi_i+\tau^{-1}S_qV_q-\tau^{-1}S_q2C_{y,q}}{1-\phi_i+\tau^{-1}S_qV_q}\right).\end{aligned}$$

where the third line uses the prior expressions linking $\alpha_{i^*}^q$, α_i^y, and $\alpha_{i^*}^y$ to V_q and $C_{y,q}$. Thus, the equilibrium value of $C_{y,q}$ is

$$\hat{C}_{y,q} = \frac{\frac{(1-\rho)\sigma_i^2}{1-\phi_i+\tau^{-1}S_q\hat{V}_q}\frac{\delta}{1-\delta\phi_i}}{1+2\tau^{-1}S_q\left(\frac{(1-\rho)\sigma_i^2}{\left(1-\phi_i+\tau^{-1}S_q\hat{V}_q\right)^2}\frac{\delta}{1-\delta\phi_i}\right)} > 0$$

where $\rho < 1$.

We now show that

$$1-\phi_i+\tau^{-1}S_q\hat{V}_q > \tau^{-1}S_q\hat{C}_{y,q},$$

which guarantees that $\hat{\alpha}_i^y > 0$. We have

$$\begin{aligned}
\hat{C}_{y,q} \quad &= \quad \frac{\delta}{1-\delta}(1-\rho)\,\sigma_i^2\hat{\alpha}_{i^*}^q\left(\hat{\alpha}_i^y-\hat{\alpha}_{i^*}^y\right) > 0 \\
&\iff \left(\hat{\alpha}_i^y-\hat{\alpha}_{i^*}^y\right) > 0 \text{ [since } \frac{\delta}{1-\delta}(1-\rho)\,\sigma_i^2\hat{\alpha}_{i^*}^q > 0] \\
&\iff \left(\frac{1-\delta}{1-\delta\phi_i}-2\hat{\alpha}_{i^*}^y\right) > 0 \text{ [since } \hat{\alpha}_i^y = \frac{1-\delta}{1-\delta\phi_i}-\hat{\alpha}_{i^*}^y] \\
&\iff 1 > \frac{2\tau^{-1}S_q\hat{C}_{y,q}}{1-\phi_i+\tau^{-1}S_q\hat{V}_q} \text{ [since } \hat{\alpha}_{i^*}^y = \frac{1-\delta}{1-\delta\phi_i}\frac{\tau^{-1}S_q\hat{C}_{y,q}}{1-\phi_i+\tau^{-1}S_q\hat{V}_q}.
\end{aligned}$$

Since $\hat{C}_{y,q} > 0$, it then follows that

$$1-\phi_i+\tau^{-1}S_q\hat{V}_q > 2\tau^{-1}S_q\hat{C}_{y,q} > \tau^{-1}S_q\hat{C}_{y,q}.$$

Finally, we show that $\gamma_{i^*}^q + (\gamma_{i^*}^{y*}-\gamma_{i*}^y) > 0$. We have

$$\gamma_{i^*}^q + (\gamma_{i^*}^{y*}-\gamma_{i*}^y) = \frac{\tau^{-1}S_q\left(V_q-2C_{y,q}\right)}{1-\phi_i+\tau^{-1}S_qV_q}.$$

Thus, it suffices to show that $0 < V_q - 2C_{y,q}$. We have

$$\begin{aligned}
[V_q-2C_{y,q}] \quad &= \quad Cov_t\left[\left(\alpha_{i^*}^q-\frac{\delta}{1-\delta}\left(\alpha_i^y-\alpha_{i^*}^y\right)\right)\left(\varepsilon_{i^*,t+1}-\varepsilon_{i,t+1}\right),\alpha_{i^*}^q\left(\varepsilon_{i^*,t+1}-\varepsilon_{i,t+1}\right)\right] \\
&= \quad \alpha_{i^*}^q\left(\alpha_{i^*}^q-\frac{\delta}{1-\delta}\left(\alpha_i^y-\alpha_{i^*}^y\right)\right)2\sigma_i^2(1-\rho) \\
&= \quad \left(\frac{1}{1-\phi_i+\tau^{-1}S_qV_q}\right)^2\left(1-\delta\left(\frac{1-\phi_i+\tau^{-1}S_q\left[V_q-2C_{y,q}\right]}{1-\delta\phi_i}\right)\right)2\sigma_i^2(1-\rho)
\end{aligned}$$

Thus, we have

$$[V_q-2C_{y,q}] = \frac{\left(\frac{1}{1-\phi_i+\tau^{-1}S_qV_q}\right)^2\frac{1-\delta}{1-\delta\phi_i}2\sigma_i^2(1-\rho)}{1+\left(\frac{1}{1-\phi_i+\tau^{-1}S_q\hat{V}_q}\right)^2\delta\left(\frac{\tau^{-1}S_q}{1-\delta\phi_i}\right)2\sigma_i^2(1-\rho)} \geq 0.$$

Furthermore, we have $[V_q-2C_{y,q}] > 0$ when $\delta < 1$ and $\lim_{\delta\to 1}[V_q-2C_{y,q}] = 0$.

A.6.2 Calculations for Case #2: $S_q = 0$ and $S_y > 0$.

General When $S_q = 0$ and $S_y > 0$, the loadings on i_t are

$$\begin{bmatrix} \alpha_i^y \\ \alpha_i^{y*} \\ \alpha_i^q \end{bmatrix} = \begin{bmatrix} \frac{1-\delta\phi_i}{1-\delta} + \tau^{-1} S_y V_y & \tau^{-1} S_y C_{y,y^*} & 0 \\ \tau^{-1} S_y C_{y,y^*} & \frac{1-\delta\phi_i}{1-\delta} + \tau^{-1} S_y V_y & 0 \\ \tau^{-1} S_y C_{y,q} & -\tau^{-1} S_y C_{y,q} & -(1-\phi_i) \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \tag{42}$$

$$= \begin{bmatrix} \left(\frac{1-\delta\phi_i}{1-\delta} + \tau^{-1} S_y V_y - \frac{\left(\tau^{-1} S_y C_{y,y*}\right)^2}{\frac{1-\delta\phi_i}{1-\delta} + \tau^{-1} S_y V_y} \right)^{-1} \\ -\frac{\tau^{-1} S_y C_{y,y}}{\frac{1-\delta\phi_i}{1-\delta} + \tau^{-1} S_y V_y} \alpha_i^y \\ -\frac{1}{1-\phi_i} + \frac{\tau^{-1} S_y C_{y,q}}{1-\phi_i} \left(\alpha_i^y - \alpha_i^{y*}\right) \end{bmatrix}$$

and the loading on i_t^* again satisfy equation (40). Since $V_y - C_{y,y*} = V_y \left(1 - Corr\left[rx_{t+1}^y, rx_{t+1}^{y*}\right]\right) > 0$, it follows that

$$\alpha_i^y - \alpha_{i^*}^y = \frac{1}{\frac{1-\delta\phi_i}{1-\delta} + \tau^{-1} S_y \left(V_y - C_{y,y*}\right)} > 0.$$

Bounding the quantities in equation (42), we have:

1. $\alpha_i^y = \left(\frac{1-\delta\phi_i}{1-\delta} + \tau^{-1} S_y V_y - \frac{\left(\tau^{-1} S_y C_{y,y*}\right)^2}{\frac{1-\delta\phi_i}{1-\delta} + \tau^{-1} S_y V_y} \right)^{-1} \in \left(0, \frac{1-\delta}{1-\delta\phi_i}\right).$

   - To show $\alpha_i^y > 0$, notice that
   $$\frac{1-\delta\phi_i}{1-\delta} + \tau^{-1} S_y V_y > 0$$
   and
   $$\left(\tau^{-1} S_y C_{y,y*}\right)^2 < \left(\tau^{-1} S_y V_y\right)^2 < \left(\frac{1-\delta\phi_i}{1-\delta} + \tau^{-1} S_y V_y\right)^2.$$
   Together these inequalities imply
   $$\frac{1-\delta\phi_i}{1-\delta} + \tau^{-1} S_y V_y - \frac{\left(\tau^{-1} S_y C_{y,y*}\right)^2}{\frac{1-\delta\phi_i}{1-\delta} + \tau^{-1} S_y V_y} > 0,$$
   confirming that $\alpha_i^y > 0$.
   - To show $\alpha_i^y < \frac{1-\delta}{1-\delta\phi_i}$, it suffices to show
   $$\tau^{-1} S_y V_y - \frac{\left(\tau^{-1} S_y C_{y,y*}\right)^2}{\frac{1-\delta\phi_i}{1-\delta} + \tau^{-1} S_y V_y} > 0.$$
   This is equivalent to
   $$\left(\frac{1-\delta\phi_i}{1-\delta} + \tau^{-1} S_y V_y\right) \tau^{-1} S_y V_y - \left(\tau^{-1} S_y C_{y,y*}\right)^2 > 0$$
   which is true since $\left(\tau^{-1} S_y C_{y,y*}\right)^2 < \left(\tau^{-1} S_y V_y\right)^2$ and $\frac{1-\delta\phi_i}{1-\delta} > 0$.

2. $\alpha_{i^*}^y = -\frac{\tau^{-1} S_y C_{y,y^*}}{\frac{1-\delta\phi_i}{1-\delta} + \tau^{-1} S_y V_y} \alpha_i^y$. Since $\alpha_i^y > 0$, we have $sign\left(\alpha_{i^*}^y\right) = sign(-C_{y,y^*})$.

3. $\alpha_i^q = -\frac{1}{1-\phi_i}+\frac{\tau^{-1}S_y C_{y,q}}{1-\phi_i}(\alpha_i^y-\alpha_{i^*}^y) \in \left(-\frac{1}{1-\phi_i}, 0\right)$ so long as (i) $C_{y,q} > 0$ and (ii) $\tau^{-1}S_y C_{y,q}\left(\alpha_i^y - \alpha_i^{y*}\right) < 1$.

   - Since $(\alpha_i^y - \alpha_{i^*}^y) > 0$, we have $\alpha_i^q > -\frac{1}{1-\phi_i}$ so long as (i) $C_{y,q} > 0$.
   - And, we have $\alpha_i^q < 0$ so long as (ii) $\tau^{-1}S_y C_{y,q}(\alpha_i^y - \alpha_{i^*}^y) < 1$.

We now explore how changes in short-term interest rates impact equilibrium expected excess returns. So long as $C_{y,q} > 0$, we have

$$\gamma_{i^*}^q \equiv \partial E_t\left[rx_{t+1}^q\right]/\partial i_t^* = (\phi_i - 1)\,\alpha_{i^*}^q + 1 = \tau^{-1}S_y C_{y,q}(\alpha_i^y - \alpha_{i^*}^y) > 0.$$

Symmetrically, we have $\gamma_i^q \equiv \partial E_t\left[rx_{t+1}^q\right]/\partial i_t = -\gamma_{i^*}^q$. Furthermore, since $\alpha_i^y < (1-\delta)/(1-\delta\phi_i)$, we have

$$\gamma_i^y \equiv \partial E_t\left[rx_{t+1}^y\right]/\partial i_t = \left(\frac{1-\delta\phi_i}{1-\delta}\right)\alpha_i^y - 1 < 0.$$

Finally, so long as $C_{y,y^*} > 0$, we have $\alpha_{i^*}^y < 0$ and thus

$$\gamma_{i^*}^y \equiv \partial E_t\left[rx_{t+1}^y\right]/\partial i_t^* = \left(\frac{1-\delta\phi_i}{1-\delta}\right)\alpha_{i^*}^y < 0.$$

Symmetrically, we have $\gamma_{i*}^{y*} \equiv \partial E_t\left[rx_{t+1}^{y*}\right]/\partial i_t^* = \gamma_i^y$ and $\gamma_i^{y*} \equiv \partial E_t\left[rx_{t+1}^{y*}\right]/\partial i_t = \gamma_{i^*}^y$.

Next, we show that $\gamma_i^y - \gamma_{i^*}^y < 0$. To see this, note that

$$\begin{aligned}\gamma_i^y - \gamma_{i^*}^y &= \left(\frac{1-\delta\phi_i}{1-\delta}\right)(\alpha_i^y - \alpha_{i^*}^y) - 1 \\ &= \left(\frac{1-\delta\phi_i}{1-\delta}\right)\alpha_i^y\left(1+\frac{\tau^{-1}S_y C_{y,y^*}}{\frac{1-\delta\phi_i}{1-\delta}+\tau^{-1}S_y V_y}\right) - 1.\end{aligned}$$

Thus, we have $\gamma_i^y - \gamma_{i^*}^y < 0$ so long as we have

$$\frac{1-\delta}{1-\delta\phi_i}\frac{\frac{1-\delta\phi_i}{1-\delta}+\tau^{-1}S_y V_y}{\frac{1-\delta\phi_i}{1-\delta}+\tau^{-1}S_y V_y+\tau^{-1}S_y C_{y,y^*}} > \alpha_i^y = \left(\frac{1-\delta\phi_i}{1-\delta}+\tau^{-1}S_y V_y - \frac{\left(\tau^{-1}S_y C_{y,y*}\right)^2}{\frac{1-\delta\phi_i}{1-\delta}+\tau^{-1}S_y V_y}\right)^{-1}.$$

One can show that this condition is equivalent to

$$\frac{1-\delta}{1-\delta\phi_i}\tau^{-1}S_y\left(V_y - C_{y,y^*}\right) > 0,$$

which is always true since $V_y > C_{y,y^*}$.

Closing the model in Case #2 when there are no supply shocks When there are no independent supply shocks—i.e., when $\sigma_{s^y}^2 = \sigma_{s^q}^2 = 0$, it is easy to confirm that we must have $C_{y,q} > 0$ and (ii) $\tau^{-1}S_y C_{y,q}(\alpha_i^y - \alpha_{i^*}^y) < 1$. Thus, the bounds noted above must indeed hold in this case. We can also show that we must have $C_{y,y^*} > 0$ in this case.

When $\sigma^2_{s^y} = \sigma^2_{s^q} = 0$, we have

$$\begin{aligned} C_{y,q} &= -\frac{\delta}{1-\delta} Cov\left[\alpha^y_i \varepsilon_{i,t+1} + \alpha^y_{i^*}\varepsilon_{i^*,t+1}, \alpha^q_{i^*}\left(\varepsilon_{i^*,t+1} - \varepsilon_{i,t+1}\right)\right] \\ &= \frac{\delta}{1-\delta}\left(1-\rho\right)\sigma^2_i \alpha^q_{i^*}\left(\alpha^y_i - \alpha^y_{i^*}\right) \\ &= \sigma^2_i \left(1-\rho\right)\frac{\delta}{1-\delta}\frac{\left(\alpha^y_i - \alpha^y_{i^*}\right)}{1-\phi_i}\left(1 - \tau^{-1} S_y C_{y,q}(\alpha^y_i - \alpha^y_{i^*})\right) \end{aligned}$$

where the last line follows from the fact that

$$\alpha^q_{i^*} = \frac{1}{1-\phi_i}\left(1 - \tau^{-1} S_y C_{y,q}(\alpha^y_i - \alpha^y_{i^*})\right).$$

Thus, given an equilibrium solution for $(\hat{\alpha}^y_i - \hat{\alpha}^y_{i^*}) > 0$, we have

$$\hat{C}_{y,q} = \frac{\sigma^2_i \left(1-\rho\right)\frac{\delta}{1-\delta}\frac{(\hat{\alpha}^y_i - \hat{\alpha}^y_{i^*})}{1-\phi_i}}{1 + \tau^{-1} S_y \sigma^2_i \left(1-\rho\right)\frac{\delta}{1-\delta}\frac{(\hat{\alpha}^y_i - \hat{\alpha}^y_{i^*})^2}{1-\phi_i}} > 0.$$

We also have

$$\tau^{-1} S_y \hat{C}_{y,q}\left(\hat{\alpha}^y_i - \hat{\alpha}^y_{i^*}\right) = \frac{\tau^{-1} S_y \sigma^2_i \left(1-\rho\right)\frac{\delta}{1-\delta}\frac{(\hat{\alpha}^y_i - \hat{\alpha}^y_{i^*})^2}{1-\phi_i}}{1 + \tau^{-1} S_y \sigma^2_i \left(1-\rho\right)\frac{\delta}{1-\delta}\frac{(\hat{\alpha}^y_i - \hat{\alpha}^y_{i^*})^2}{1-\phi_i}} < 1.$$

We show that we must have $C_{y,y^*} > 0$. We have

$$\begin{aligned} C_{y,y^*} &= \left(\frac{\delta}{1-\delta}\right)^2 Cov\left[\alpha^y_i \varepsilon_{i,t+1} + \alpha^y_{i^*}\varepsilon_{i^*,t+1}, \alpha^y_{i^*}\varepsilon_{i,t+1} + \alpha^y_i \varepsilon_{i^*,t+1}\right] \\ &= \left(\frac{\delta}{1-\delta}\right)^2 \left[2\alpha^y_i \alpha^y_{i^*}\sigma^2_i + \left(\alpha^y_i\right)^2 \rho\sigma^2_i + \left(\alpha^y_{i^*}\right)^2 \rho\sigma^2_i\right] \\ &= \left(\frac{\delta}{1-\delta}\right)^2 \left(\alpha^y_i\right)^2 \sigma^2_i \left[\rho\left(\frac{\tau^{-1} S_y C_{y,y^*}}{\frac{1-\delta\phi_i}{1-\delta} + \tau^{-1} S_y V_y}\right)^2 - 2\frac{\tau^{-1} S_y C_{y,y^*}}{\frac{1-\delta\phi_i}{1-\delta} + \tau^{-1} S_y V_y} + \rho\right]. \end{aligned}$$

Suppose that $C_{y,y^*} < 0$. If $C_{y,y^*} < 0$, then this equation implies that $C_{y,y^*} > 0$ so long as $\rho > 0$: a contradiction. Thus, when $\rho > 0$, we must have $C_{y,y^*} > 0$ in equilibrium. Of course, when $\rho = 0$, we have $C_{y,y^*} = 0$ in equilibrium.

Finally, we show that $\gamma^q_{i^*} + (\gamma^{y*}_{i^*} - \gamma^y_{i^*}) \geq 0$. We have

$$\gamma^q_{i^*} + (\gamma^{y*}_{i^*} - \gamma^y_{i^*}) = \tau^{-1} S_y C_{y,q}(\alpha^y_i - \alpha^y_{i^*}) + \left(\frac{1-\delta\phi_i}{1-\delta}\right)(\alpha^y_i - \alpha^y_{i^*}) - 1 = \frac{\tau^{-1} S_y \left(C_{y,y*} + C_{y,q} - V_y\right)}{\frac{1-\delta\phi_i}{1-\delta} + \tau^{-1} S_y \left(V_y - C_{y,y*}\right)}$$

Since $V_y > C_{y,y*}$, it suffices to show that $0 < C_{y,y*} + C_{y,q} - V_y$. We have

$$\begin{aligned}
& [C_{y,y*} + C_{y,q} - V_y] \\
= & \; Cov_t\left[\left(\alpha_{i^*}^q - \frac{\delta}{1-\delta}\left(\alpha_i^y - \alpha_{i^*}^y\right)\right)\left(\varepsilon_{i^*,t+1} - \varepsilon_{i,t+1}\right), -\frac{\delta}{1-\delta}\left(\alpha_i^y \varepsilon_{i,t+1} + \alpha_{i^*}^y \varepsilon_{i^*,t+1}\right)\right] \\
= & \left(\alpha_{i^*}^q - \frac{\delta}{1-\delta}\left(\alpha_i^y - \alpha_{i^*}^y\right)\right)\left(\frac{\delta}{1-\delta}\left(\alpha_i^y - \alpha_{i^*}^y\right)\right)(1-\rho)\,\sigma_i^2 \\
= & \; (1-\rho)\,\sigma_i^2 \frac{\delta(1-\delta)}{1-\phi_i} \frac{1 - \tau^{-1} S_y \left[C_{y,y*} + C_{y,q} - V_y\right]}{\left(1 - \delta\phi_i + (1-\delta)\,\tau^{-1} S_y \left(V_y - C_y\right)\right)^2}.
\end{aligned}$$

Thus, we have

$$[C_{y,y*} + C_{y,q} - V_y] = \frac{(1-\rho)\,\sigma_i^2 \frac{\delta(1-\delta)}{1-\phi_i} \frac{1}{(1-\delta\phi_i + (1-\delta)\tau^{-1} S_y (V_y - C_y))^2}}{1 + (1-\rho)\,\sigma_i^2 \frac{\delta(1-\delta)}{1-\phi_i} \frac{\tau^{-1} S_y}{(1-\delta\phi_i + (1-\delta)\tau^{-1} S_y (V_y - C_y))^2}} \geq 0.$$

Thus, we have $[\gamma_{i^*}^q + (\gamma_{i^*}^{y*} - \gamma_i^{y^*})] > 0$ when $\delta < 1$ and $\lim_{\delta \to 1}[\gamma_{i^*}^q + (\gamma_{i^*}^{y*} - \gamma_i^{y^*})] = 0$.

A.6.3 Regression calculations

Fama (1984) regressions Consider the excess returns on FX carry trade. Suppose we estimate the following time-series forecasting regression

$$rx_{t+1}^q = \alpha_q + \beta_q \left(i_t^* - i_t\right) + \xi_{t+1}^q.$$

We have

$$E_t\left[rx_{t+1}^q\right] = \gamma_{i^*}^q \times \left(i_t^* - i_t\right) + \gamma_{s^y}^q \times \left(s_t^y - s_t^{y^*}\right) + \gamma_{s^q}^q \times s_t^q,$$

where $\gamma_f^q \equiv \partial E_t\left[rx_{t+1}^q\right]/\partial f_t$ for $f_t \in (i_t, i_t^*, s_t^y, s_t^{y*}, s_t^q)$ and we have made use of the fact that $\gamma_i^q = -\gamma_{i^*}^q$ and $\gamma_{s^{y*}}^q = -\gamma_{s^y}^q$. Because independent movements in asset supply $(s_t^y, s_t^{y^*}, s_t^q)$ are orthogonal to the interest rate differential by assumption, it follows that

$$\beta_q = \frac{Cov\left[rx_{t+1}^q, i_t^* - i_t\right]}{Var\left[i_t^* - i_t\right]} = \frac{Cov\left[E_t\left[rx_{t+1}^q\right], i_t^* - i_t\right]}{Var\left[i_t^* - i_t\right]} = \gamma_{i^*}^q = \frac{\partial E_t\left[rx_{t+1}^q\right]}{\partial i_t^*} > 0.$$

Thus, in either Case #1 or Case #2, the extended model matches Fama's (1984) finding that the expected returns on the borrow-home-lend-abroad FX carry trade are high when the foreign-minus-domestic interest rate differential is high.

Lustig, Stathopoulos, and Verdelhan (2019) regressions Consider the regression

$$rx_{t+1}^q + \left(rx_{t+1}^{y*} - rx_{t+1}^y\right) = \alpha_{q,lt} + \beta_{q,lt} \times \left(i_t^* - i_t\right) + \xi_{t+1}^{q,lt}. \tag{43}$$

We want to show that $0 < \beta_{q,lt} < \beta_q$ as in Lustig, Stathopoulos, and Verdelhan (2019). Since

$$\beta_{q,lt} = \beta_q + \beta_{y^*-y} = \gamma_{i^*}^q + \left(\gamma_{i^*}^{y*} - \gamma_i^{y*}\right),$$

it suffices to show that $-\beta_q < \beta_{y^*-y} < 0$, where β_{y^*-y} is the coefficient from the regression

$$rx_{t+1}^{y*} - rx_{t+1}^y = \alpha_{y^*-y} + \beta_{y^*-y} \times \left(i_t^* - i_t\right) + \xi_{t+1}^{y^*-y}.$$

We first consider β_{y^*-y}. We have

$$E_t\left[rx^{y*}_{t+1} - rx^{y}_{t+1}\right] = \left(\gamma^{y*}_{i^*} - \gamma^{y*}_{i}\right)\left(i^*_t - i_t\right) + \left(\gamma^{y*}_{s^{y*}} - \gamma^{y*}_{s^y}\right)\left(s^{y*}_t - s^y_t\right) + 2\gamma^{y*}_{s^q} s^q_t.$$

It follows that

$$\begin{aligned} \beta_{y^*-y} &= \frac{Cov\left[rx^{y*}_{t+1} - rx^{y}_{t+1}, i^*_t - i_t\right]}{Var\left[i^*_t - i_t\right]} \\ &= \frac{\partial E_t[rx^{y^*}_{t+1}]}{\partial i^*_t} - \frac{\partial E_t[rx^{y^*}_{t+1}]}{\partial i_t} = \gamma^{y*}_{i^*} - \gamma^{y*}_{i}. \end{aligned}$$

We have $\gamma^{y*}_{i^*} - \gamma^{y*}_{i} < 0$ under either Case #1 or #2. This is trivial under Case #1 since in that case $\gamma^{y*}_{i^*} < 0$ and $\gamma^{y*}_{i} > 0$. It is also negative under Case #2 since in that case we have $\gamma^{y}_{i} - \gamma^{y}_{i^*} < 0$ (even though we have $\gamma^{y*}_{i^*} < 0$ and $\gamma^{y*}_{i} < 0$). It follows that $\beta_{y^*-y} < 0$ and, therefore, that $\beta_{q,lt} = \beta_q + \beta_{y^*-y} < \beta_q$.

We now show that $\beta_{q,lt} > 0$ or, equivalently, $-\beta_q < \beta_{y^*-y}$. We have

$$-\beta_q = -\gamma^{q}_{i^*} < \gamma^{y*}_{i^*} - \gamma^{y*}_{i} = \beta_{y^*-y}.$$

It suffices to show that

$$0 < \gamma^{q}_{i^*} + \left(\gamma^{y*}_{i^*} - \gamma^{y*}_{i}\right).$$

As shown above, in both Case #1 and Case #2 we have $\gamma^{q}_{i^*} + (\gamma^{y*}_{i^*} - \gamma^{y^*}_{i}) > 0$ when $\delta < 1$ and $\lim_{\delta\to 1}[\gamma^{q}_{i^*} + (\gamma^{y*}_{i^*} - \gamma^{y^*}_{i})] = 0$.

Campbell and Shiller (1991) regressions Consider the excess returns on the yield curve carry trade. Suppose we estimate the following time-series forecasting regression

$$rx^{y}_{t+1} = \alpha_y + \beta_y\left(y_t - i_t\right) + \xi^{y}_{t+1}.$$

We have

$$E_t\left[rx^{y}_{t+1}\right] = E\left[rx^{y}_{t+1}\right] + \gamma^{y}_{i}\left(i_t - \bar{\imath}\right) + \gamma^{y}_{i^*}\left(i^*_t - \bar{\imath}\right) + \gamma^{y}_{s^y}\left(s^y_t - \bar{s}^y\right) + \gamma^{y}_{s^{y*}}\left(s^{y*}_t - \bar{s}^y\right) + \gamma^{y}_{s^q} s^q_t$$

where $\gamma^{y}_{f} \equiv \partial E_t\left[rx^{y}_{t+1}\right]/\partial f_t$ for $f_t \in (i_t, i^*_t, s^y_t, s^{y*}_t, s^q_t)$. The term spread is given by

$$\left(y_t - i_t\right) = \alpha^{y}_{0} + \left(\alpha^{y}_{i} - 1\right)\left(i_t - \bar{\imath}\right) + \alpha^{y}_{i^*}\left(i^*_t - \bar{\imath}\right) + \alpha^{y}_{s^y}\left(s^y_t - \bar{s}^y\right) + \alpha^{y}_{s^{y*}}\left(s^{y*}_t - \bar{s}^y\right) + \alpha^{y}_{s^q} s^q_t.$$

We have $\beta_y = Cov\left[y_t - i_t, rx^{y}_{t+1}\right]/Var\left[y_t - i_t\right]$. Thus, we have

$$\begin{aligned} \beta_y &\propto Cov\left[y_t - i_t, rx^{y}_{t+1}\right] \\ &= \left[\gamma^{y}_{i}\left(\alpha^{y}_{i} - 1\right) + \rho\left(\alpha^{y}_{i} - 1\right)\gamma^{y}_{i^*} + \rho\alpha^{y}_{i^*}\gamma^{y}_{i} + \alpha^{y}_{i^*}\gamma^{y}_{i^*}\right]\frac{\sigma^2_i}{1-\phi^2_i} \\ &\quad + \left[\left(\gamma^{y}_{s^y}\alpha^{y}_{s^y} + \gamma^{y}_{s^{y*}}\alpha^{y}_{s^{y*}}\right)\frac{\sigma^2_{s^y}}{1-\phi^2_{s_y}} + \left(\gamma^{y}_{s^q}\alpha^{y}_{s^q}\right)\frac{\sigma^2_{s^q}}{1-\phi^2_{s_q}}\right] \end{aligned}$$

We wish to show that $\beta_y > 0$. For any supply factor $f_t \in (s^y_t, s^{y*}_t, s^q_t)$, we have $sign(\gamma^{y}_{f}) = sign(\alpha^{y}_{f})$. It thus follows that $(\gamma^{y}_{s^y}\alpha^{y}_{s^y} + \gamma^{y}_{s^{y*}}\alpha^{y}_{s^{y*}}) > 0$ and $(\gamma^{y}_{s^q}\alpha^{y}_{s^q}) > 0$. Thus, the second term is square brackets is always positive. Thus, to prove that $\beta_y > 0$, it suffices to show that the first term in square brackets is positive—i.e., to show that

$$(\alpha_i^y - 1)(\gamma_i^y + \rho\gamma_{i*}^y) + \alpha_{i*}^y(\rho\gamma_i^y + \gamma_{i*}^y) > 0.$$

We prove this inequality separately in Case #1 and Case #2 below under the simplifying assumption that $\sigma_{s^y}^2 = \sigma_{s^q}^2 = 0$. Finally, letting

$$rx_{t+1}^{y*} = \alpha_{y*} + \beta_{y*}(y_t^* - i_t^*) + \xi_{t+1}^{y*},$$

we have $\beta_{y*} = \beta_y$ by symmetry.

Case #1 In Case #1 where $S_q > 0$ and $S_y = 0$, we have $\gamma_i^y = -\gamma_{i*}^y < 0$. By definition, we have

$$\gamma_i^y = \frac{1-\delta\phi_i}{1-\delta}\alpha_i^y - 1 \text{ or } \alpha_i^y = \frac{1-\delta}{1-\delta\phi_i}(1+\gamma_i^y).$$

We also have

$$\alpha_{i*}^y = \frac{1-\delta}{1-\delta\phi_i}\gamma_{i*}^y = -\frac{1-\delta}{1-\delta\phi_i}\gamma_i^y.$$

Substituting these expressions for α_i^y, α_{i*}^y, and γ_{i*}^y, we obtain

$$\begin{aligned}
& (\alpha_i^y - 1)(\gamma_i^y + \rho\gamma_{i*}^y) + \alpha_{i*}^y(\rho\gamma_i^y + \gamma_{i*}^y) \\
= \quad & (1-\rho)\left[\left(\frac{1-\delta}{1-\delta\phi_i} - 1\right)\gamma_i^y + 2\frac{1-\delta}{1-\delta\phi_i}(\gamma_i^y)^2\right] > 0,
\end{aligned}$$

where the inequality follows because $(1-\delta)/(1-\delta\phi_i) < 1$ and $\gamma_i^y < 0$.

Case #2 In Case #2 where $S_q = 0$ and $S_y > 0$, we have $\gamma_i^y < 0$. Since we have

$$\alpha_{i*}^y = -c\alpha_i^y < 0$$

for some constant $c > 0$, we have

$$\alpha_i^y = \frac{1-\delta}{1-\delta\phi_i}(1+\gamma_i^y) \text{ and } \alpha_{i*}^y = -c\frac{1-\delta}{1-\delta\phi_i}(1+\gamma_i^y).$$

Since $\alpha_i^y > 0$ and $\gamma_i^y < 0$, we also have $0 < (1+\gamma_i^y) < 1$. Finally, we have

$$\gamma_{i*}^y = \frac{1-\delta\phi_i}{1-\delta}\alpha_{i*}^y = -c(1+\gamma_i^y) < 0.$$

Substituting these expressions for α_i^y, α_{i*}^y, and γ_{i*}^y, we obtain

$$\begin{aligned}
& (\alpha_i^y - 1)(\gamma_i^y + \rho\gamma_{i*}^y) + \alpha_{i*}^y(\rho\gamma_i^y + \gamma_{i*}^y) \\
= \quad & \overbrace{\left(\frac{1-\delta}{1-\delta\phi_i}(1+\gamma_i^y) - 1\right)}^{<0}\overbrace{(\gamma_i^y - \rho c(1+\gamma_i^y))}^{<0} + \overbrace{\left(-c\frac{1-\delta}{1-\delta\phi_i}(1+\gamma_i^y)\right)}^{<0}\overbrace{(\rho\gamma_i^y - c(1+\gamma_i^y))}^{<0} > 0.
\end{aligned}$$

A.7 Contrast with frictionless asset-pricing models

In this Appendix, we contrast the results from our baseline model in Section 3 with those implied by frictionless, consumption-based asset pricing models. Consider a frictionless asset-pricing model featuring complete international financial markets, but imperfect risk sharing between the home and foreign countries. Since financial markets are complete, the stochastic discount factor is unique, implying:

$$M^*_{t+1} = M_{t+1} \left(Q_{t+1}/Q_t \right). \tag{44}$$

where Q_t is the foreign exchange rate, M_{t+1} is stochastic discount factor (SDF) that price all returns in domestic currency, and M^*_{t+1} is discount factor pricing all returns in formal currency (Backus, Foresi, Telmer [2001]).

Taking logs we find:

$$q_{t+1} - q_t = m^*_{t+1} - m_{t+1}. \tag{45}$$

Thus, frictionless theories imply that foreign currency appreciates in bad times for foreign agents where m^*_{t+1} is high and depreciates in bad times for domestic agents when m_{t+1} is high. These exchange rate dynamics make domestic assets risky for foreign agents and vice versa, rationalizing imperfect international risk sharing even with complete financial markets.

As shown in Table 6, consumption-based theories typically imply that foreign interest rates decline in bad times for foreign agents, so standard uncovered-interest-rate-parity (UIP) logic pushes foreign currency toward depreciating in bad times for foreign agents. However, by construction, this UIP effect needs to more than fully offset in consumption-based models by either a temporary appreciation of foreign currency (i.e., by news that the expected returns on foreign currency will be lower going forward, perhaps, because $E_t \left[rx^q_{t+1} \right]$ is increasing in $(i^*_t - i_t)$) or by a permanent appreciation (i.e., by an innovation to a random walk component of the exchange rate).[11] Thus, many leading consumption-based models imply

$$Cov_t \left[\Delta q_{t+1}, \Delta \left(i^*_{t+1} - i_{t+1} \right) \right] = Cov_t \left[rx^q_{t+1}, i^*_{t+1} - i_{t+1} \right] < 0. \tag{46}$$

By contrast, in our theory as in the data, we have $Cov_t \left[\Delta q_{t+1}, \Delta \left(i^*_{t+1} - i_{t+1} \right) \right] > 0$.

Assuming that both the foreign and domestic SDFs are log-normally distributed, we have

$$E_t \left[rx^q_{t+1} \right] = E_t \left[q_{t+1} - q_t + (i^*_t - i_t) \right] = \frac{1}{2} \left(\sigma^2_t [m_{t+1}] - \sigma^2_t \left[m^*_{t+1} \right] \right), \tag{47}$$

which follows from the facts that $q_{t+1} - q_t = m^*_{t+1} - m_{t+1}$, $i_t = -E_t[m_{t+1}] - \sigma^2_t[m_{t+1}]/2$, and $i^*_t = -E_t[m^*_{t+1}] - \sigma^2_t[m^*_{t+1}]/2$. Thus, the expected excess return on foreign currency is one half the difference between the conditional variances of the domestic and foreign log SDFs. In other words, foreign currency risk premium will be high when domestic agents are more risk averse than foreign agents or when domestic agents are exposed to greater macroeconomic risk.

[11] We have $q_{t+1} = -\sum_{j=1}^{T} (m^*_{t+1+j} - m_{t+1+j}) + q_{t+T}$. Letting $E_{t+1}\left[q_{t+\infty}\right] \equiv \lim_{T\to\infty} E_{t+1}\left[q_{t+T}\right]$ and taking expectations and the limit as $T \to \infty$, we obtain $q_{t+1} = -\sum_{j=1}^{\infty} E_{t+1}[m^*_{t+1+j} - m_{t+1+j}] + E_{t+1}\left[q_{t+\infty}\right] = \sum_{j=0}^{\infty} E_{t+1}[i^*_{t+1+j} - i_{t+1+j} - rx^q_{t+2+j}] + E_{t+1}\left[q_{t+\infty}\right]$. Since

$$(E_{t+1} - E_t)\, q_{t+1} = \overbrace{\textstyle\sum_{j=0}^{\infty} (E_{t+1} - E_t) \left[i^*_{t+1+j} - i_{t+1+j} \right]}^{\mathcal{N}^{i^*-i}_{t+1}} - \overbrace{\textstyle\sum_{j=0}^{\infty} (E_{t+1} - E_t) \left[rx^q_{t+2+j} \right]}^{\mathcal{N}^{rx^q}_{t+1}} + \overbrace{(E_{t+1} - E_t) \left[q_{t+\infty} \right]}^{\mathcal{N}^{q_\infty}_t}$$

unexpected movements in exchange rates must either reflect news about the future interest rate differentials ($\mathcal{N}^{i^*-i}_{t+1}$), news about future excess returns on foreign exchange ($\mathcal{N}^{rx^q}_{t+1}$), or permanent news about the long-run level of foreign currency ($\mathcal{N}^{q_\infty}_t$).

Similarly, assuming the local-currency excess returns on long-term bonds are jointly log-normal, we have:

$$E_t[rx^y_{t+1}] + \frac{1}{2}\sigma^2_t[rx^y_{t+1}] = -Corr_t[rx^y_{t+1}, m_{t+1}]\sigma_t[rx^y_{t+1}]\sigma_t[m_{t+1}], \tag{48a}$$

$$E_t[rx^{y^*}_{t+1}] + \frac{1}{2}\sigma^2_t[rx^{y^*}_{t+1}] = -Corr_t[rx^{y^*}_{t+1}, m^*_{t+1}]\sigma_t[rx^{y^*}_{t+1}]\sigma_t\left[m^*_{t+1}\right]. \tag{48b}$$

Consumption-based models almost always imply that $Corr_t[rx^y_{t+1}, m_{t+1}] > 0$ and $Corr_t[rx^{y^*}_{t+1}, m^*_{t+1}] > 0$—i.e., long-term domestic (foreign) bonds are an attractive hedge for domestic (foreign) investors. The idea is that domestic interest rates typically decline when the domestic agents' marginal value of financial wealth is unexpectedly high (e.g., because the SDF is persistent or because the volatility of the SDF rises in bad times), leading the prices of long-term domestic bonds to rise in these states of the world.

In our model, $E_t\left[rx^q_{t+1}\right]$ is negatively related to $E_t[rx^{y^*}_{t+1} - rx^y_{t+1}]$—i.e., the expected excess returns on foreign exchange are decreasing in the foreign-minus-domestic term premium. What do leading consumption-based model imply? In modern consumption-based models, the main reason expected returns fluctuate over time is because the conditional volatilities of SDFs ($\sigma_t[m_{t+1}]$ and $\sigma_t[m^*_{t+1}]$) vary over time—e.g., due to time-varying risk aversion as in habit formation models (Campbell and Cochrane [1999]), time-varying consumption volatility as in long-run risks models (Bansal and Yaron [2004]), or a time-varying probability of a rare economic disaster (Gabaix [2012] and Wachter [2013]). Thus, since $Corr_t[rx^y_{t+1}, m_{t+1}] > 0$, an increase in $\sigma_t[m_{t+1}]$ raises $E_t[rx^q_{t+1}]$, but reduces $E_t[rx^y_{t+1}]$—i.e., $Corr\left(E_t[rx^q_{t+1}], E_t[rx^y_{t+1}]\right) < 0$. By contrast, in our model, $E_t[rx^q_{t+1}]$ tends to be high at the same time that $E_t[rx^y_{t+1}]$ is also high—i.e., $Corr\left(E_t[rx^q_{t+1}], E_t[rx^y_{t+1}]\right) > 0$. Symmetrically, since $Corr_t[rx^{y^*}_{t+1}, m^*_{t+1}] > 0$, an increase in $\sigma_t[m^*_{t+1}]$ reduces $E_t[rx^q_{t+1}]$ and also reduces $E_t[rx^{y^*}_{t+1}]$—i.e., $Corr(E_t[rx^q_{t+1}], E_t[rx^{y^*}_{t+1}]) > 0$. By contrast, in our model, we have $Corr(E_t[rx^q_{t+1}], E_t[rx^{y^*}_{t+1}]) < 0$.

This crucial difference stems from two differences between our theory and standard frictionless theories. First, we assume that the global rates market is partially segmented from the broader capital markets as well as from ultimate consumption. As a result, long-term bonds are potentially risky for the specialized bond investors who are the relevant marginal holders of long-term bonds. Second, in consumption-based models, the realized returns on foreign currency are positively correlated with those on long-term foreign bonds and negatively correlated with those on domestic bonds. By contrast, in our theory as in the data, the realized returns on foreign currency are negatively correlated with those on long-term foreign bonds and positively correlated with those on domestic bonds.

To see this juxtaposition starkly, suppose $\sigma^2_t[rx^y_{t+1}] = \sigma^2_t[rx^{y^*}_{t+1}] = \sigma^2_y$ and $Corr_t[rx^y_{t+1}, m_{t+1}] = Corr_t[rx^{y^*}_{t+1}, m^*_{t+1}] = \varrho_{y,m} > 0$ are constant over time, so

$$E_t[rx^y_{t+1}] + \frac{1}{2}\sigma^2_y = -\varrho_{y,m}\sigma_y\sigma_t[m_{t+1}], \tag{49a}$$

$$E_t[rx^{y^*}_{t+1}] + \frac{1}{2}\sigma^2_y = -\varrho_{y,m}\sigma_y\sigma_t[m^*_{t+1}]. \tag{49b}$$

Thus, all time-series variation in foreign and domestic bond risk premia is driven by time-variation in the conditional volatility of the domestic and foreign SDFs. However, this implies that

$$E_t[rx^{y*}_{t+1} - rx^y_{t+1}] = \varrho_{y,m}\sigma_y\left(\sigma_t[m_{t+1}] - \sigma_t[m^*_{t+1}]\right). \tag{50}$$

Using Eq. (47), we find that:

$$E_t\left[rx_{t+1}^q\right] = \overbrace{\left[\frac{\sigma_t[m_{t+1}] + \sigma_t[m_{t+1}^*]}{2\varrho_{y,m}\sigma_y}\right]}^{>0} \times E_t[rx_{t+1}^{y*} - rx_{t+1}^y]. \tag{51}$$

Thus, most consumption-based theories predict a positive relationship between FX risk premia and the difference between foreign and domestic term premia. By contrast, as emphasized in Section 3, our theory implies a negative relationship between FX risk premia and the difference between foreign and domestic bond risk premia.

Turning to the expected return to the long-term FX trade, consumption-based models in this class imply that the expected returns on the long-term carry trade are greater in magnitude than those on the short-term FX trade:

$$E_t[rx_{t+1}^q + (rx_{t+1}^{y^*} - rx_{t+1}^y)] = \overbrace{\left(1 + \frac{2\varrho_{y,m}\sigma_y}{\sigma_t[m_{t+1}] + \sigma_t\left[m_{t+1}^*\right]}\right)}^{>1} \times E_t\left[rx_{t+1}^q\right]. \tag{52}$$

By contrast, our model is consistent with the evidence in that the return on the long-term FX trade are smaller than those on the standard, short-term FX trade.

B Deviations from covered-interest-rate parity

To model deviations from covered-interest rate parity (CIP), we make two assumptions.

1. We assume that the only market participants who can engage in riskless CIP arbitrage trades—i.e., borrowing at the synthetic domestic short rate to lend at the cash domestic short rate—are a set of global banks who face non-risk-based balance sheet constraints.

2. We assume that risk-averse bond investors—who are either domiciled at home or abroad—must use FX forwards if they want to make FX-hedged investments in non-local long-term bonds. This is equivalent to saying that bond investors cannot directly borrow (i.e., obtain "cash" funding) in their non-local currency.

We assume half of all global bond investors are domiciled in the home country and half are domiciled in the foreign country. Both domestic and foreign bond investors have mean-variance preferences over one-period-ahead wealth and a risk tolerance of τ in domestic currency terms.[12] Investors differ only in terms of the returns they can earn because of CIP violations.

The vector of excess returns from t to $t+1$ to be endogenized becomes $\mathbf{rx}_{t+1} \equiv [rx_{t+1}^y, rx_{t+1}^{y*}, rx_{t+1}^q, x_t^{cip}]'$ and the vector of exogenous supplies becomes $\mathbf{s}_t \equiv [s_t^y, s_t^{y*}, s_t^q, s_t^{cip}]'$.

FX-hedged returns Consider a domestic investor taking a forward FX-hedged position at time t in a risky foreign assets with returns R_{t+1}^*. At time t, the investor converts 1 unit of domestic currency into $1/Q_t$ units of foreign currency. Suppose the investor sells forward $H_t = I_t^*$ units of foreign currency at the forward price F_t^q (this expression is valid for any H_t, but setting $H_t = I_t^* = \exp(i_t^*)$ is convenient).

[12] Thus, at time t, the risk tolerance of foreign bond investors is τ/Q_t in foreign currency terms, which corresponds to a risk tolerance of τ in domestic currency terms.

Then, the FX-hedged return return in domestic currency on the risky asset is

$$R^*_{H,t+1} = \frac{F^q_t I^*_t + (R^*_{t+1} - I^*_t)Q_{t+1}}{Q_t} = \frac{F^q_t}{Q_t} R^*_{t+1} + \overbrace{\frac{(Q_{t+1} - F_t)(R^*_{t+1} - I^*_t)}{Q_t}}^{\text{Basis risk}}.$$

Thus, the FX-hedged return includes a basis risk term that reflect the product of the excess return on foreign currency and the local-currency excess return on the risky asset.

We now assume that $F^q_t = (Q_t I_t)/(I^*_t X^{cip}_t)$ or $f^q_t = q_t - (i^*_t - i_t) - x^{cip}_t$ where $X^{cip}_t = \exp(x^{cip}_t)$. Thus, we have

$$R^*_{H,t+1} = \frac{F^q_t I^*_t + (R^*_{t+1} - I^*_t)Q_{t+1}}{Q_t} = \frac{I_t}{I^*_t X^{cip}_t} R^*_{t+1} + \left(\frac{Q_{t+1}}{Q_t} - \frac{I_t}{I^*_t}\frac{1}{X^{cip}_t}\right)(R^*_{t+1} - I^*_t).$$

Using a Taylor series expansion about the point where the realized basis risk term is zero, the log-hedged return is

$$r^*_{H,t+1} \approx [i_t + (r^*_{t+1} - i^*_t) - x^{cip}_t] + \left[\frac{(I^*_t X^{cip}_t Q_{t+1}/Q_t - I_t)(R^*_{t+1} - I^*_t)}{I_t R^*_{t+1}}\right].$$

The second term in square braces can be well approximated as

$$\frac{(I^*_t X^{cip}_t Q_{t+1}/Q_t - I_t)(R^*_{t+1} - I^*_t)}{I_t R^*_{t+1}} \approx (rx^q_{t+1} + x^{cip}_t) \times rx^*_{t+1}$$

where $rx^*_{t+1} = \ln(R^*_{t+1}) - i^*_t$. Thus, we have

$$rx^*_{H,t+1} \approx rx^*_{t+1} - x^{cip}_t + (rx^q_{t+1} + x^{cip}_t) \times rx^*_{t+1}.$$

We neglect this second basis risk term in our theoretical calculations in Section 4. Intuitively, this amounts to assuming that investors are regularly rebalancing their FX hedges. Formally, consider

$$E_t[(rx^q_{t+1} + x^{cip}_t) \times rx^*_{t+1}] = (E_t[rx^q_{t+1}] + x^{cip}_t) \times E_t[rx^*_{t+1}] + Cov_t[rx^q_{t+1}, rx^*_{t+1}].$$

If we let dt denote the return horizon, then $(E_t[rx^q_{t+1}] + x^{cip}_t) \times E_t[rx^*_{t+1}]$ will be of order $(dt)^2$ and $Cov_t[rx^q_{t+1}, rx^*_{t+1}]$ will be of order dt. Thus, in the continuous-time limit in which FX-hedges are continuously rebalanced ($dt \to 0$), the $(dt)^2$ terms vanish and we will only be left with a constant covariance term.

Domestic bond investors Domestic bond investors can obtain a riskless return of i_t from t to $t+1$ by investing in short-term domestic bonds. They can buy long-term domestic bonds, earning an excess return of rx^y_{t+1}; they can take FX-hedged positions in long-term foreign bonds, generating an excess return of $rx^{y^*}_{t+1} - x^{cip}_t$; and they can make forward investments in foreign currency, earning an excess return of $rx^q_{t+1} + x^{cip}_t$. In effect, domestic investors only have access to excess returns $\mathbf{rx}_{D,t+1} = [rx^y_{t+1}, rx^{y^*}_{t+1} - x^{cip}_t, rx^q_{t+1} + x^{cip}_t]'$.

Suppose domestic bond investors have a demand $h^y_{D,t}$ for domestic long-term bonds, $h^{y*}_{D,t}$ for FX-hedged investments in long-term foreign bonds, $h^q_{D,t}$ for forward investment in foreign currency. Thus, their positions in the four underlying long-short trades (domestic bonds, foreign bonds, cash FX investment, and CIP arbitrage) are

$$\mathbf{d}_{D,t} = \begin{bmatrix} d^{y}_{D,t} \\ d^{y*}_{D,t} \\ d^{q}_{D,t} \\ d^{x}_{D,t} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} h^{y}_{D,t} \\ h^{y*}_{D,t} \\ h^{q}_{D,t} \end{bmatrix} = \mathbf{H}_D \mathbf{h}_{D,t}.$$

Domestic bond investors choose their demands $\mathbf{h}_{D,t}$ over excess returns

$$\mathbf{rx}_{D,t+1} = \begin{bmatrix} rx^{y}_{t+1} \\ rx^{y^*}_{t+1} - x^{cip}_{t} \\ rx^{q}_{t+1} + x^{cip}_{t} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} rx^{y}_{t+1} \\ rx^{y^*}_{t+1} \\ rx^{q}_{t+1} \\ x^{cip}_{t} \end{bmatrix} = \mathbf{H}'_D \mathbf{rx}_{t+1}.$$

Thus, they solve

$$\max_{\mathbf{h}_{D,t}} \left\{ \mathbf{h}'_{D,t} E_t \left[\mathbf{rx}_{D,t+1}\right] - \frac{\tau}{2} \mathbf{h}'_{D,t} Var_t \left[\mathbf{rx}_{D,t+1}\right] \mathbf{h}_{D,t} \right\}.$$

The solution is

$$\mathbf{h}_{D,t} = \tau \left(Var_t \left[\mathbf{rx}_{D,t+1}\right]\right)^{-1} E_t \left[\mathbf{rx}_{D,t+1}\right] = \tau \left(\mathbf{H}'_D Var_t \left[\mathbf{rx}_{t+1}\right] \mathbf{H}_D\right)^{-1} \mathbf{H}'_D E_t \left[\mathbf{rx}_{t+1}\right],$$

implying that

$$\mathbf{d}_{D,t} = \mathbf{H}_D \mathbf{h}_{D,t} = \tau \mathbf{H}_D \left(\mathbf{H}'_D Var_t \left[\mathbf{rx}_{t+1}\right] \mathbf{H}_D\right)^{-1} \mathbf{H}'_D E_t \left[\mathbf{rx}_{t+1}\right].$$

Note that

$$\begin{aligned} \mathbf{H}'_D Var_t \left[\mathbf{rx}_{t+1}\right] \mathbf{H}_D &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} V_y & C_{y,y*} & C_{y,q} & 0 \\ C_{y,y*} & V_y & -C_{y,q} & 0 \\ C_{y,q} & -C_{y,q} & V_q & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} V_y & C_{y,y*} & C_{y,q} \\ C_{y,y*} & V_y & -C_{y,q} \\ C_{y,q} & -C_{y,q} & V_q \end{bmatrix} \equiv \mathbf{V}. \end{aligned}$$

Thus, CIP basis doesn't affect the risk of their investments; just the expected returns on their investments. This implies that

$$\mathbf{d}_{D,t} = \tau \mathbf{H}_D \mathbf{V}^{-1} \mathbf{H}'_D E_t \left[\mathbf{rx}_{t+1}\right].$$

Foreign bond generalists Foreign bond investors are the mirror image of domestic investors. Foreign investors have access to excess returns $\mathbf{rx}_{F,t+1} = [rx^{y}_{t+1} + x^{cip}_{t}, rx^{y^*}_{t+1}, rx^{q}_{t+1} + x^{cip}_{t}]'$. Suppose that foreign bond investors have a demand $h^{y}_{F,t}$ for FX-hedged domestic long-term bonds, $h^{y*}_{F,t}$ for foreign long-term bonds, $h^{q}_{F,t}$ for forward-investment in FX. So their positions in the four pure long-short trades are

$$\mathbf{d}_{F,t} = \begin{bmatrix} d^{y}_{F,t} \\ d^{y*}_{F,t} \\ d^{q}_{F,t} \\ d^{x}_{F,t} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} h^{y}_{F,t} \\ h^{y*}_{F,t} \\ h^{q}_{F,t} \end{bmatrix} = \mathbf{H}_F \mathbf{h}_{F,t}.$$

These foreign bond investors choose their demands $\mathbf{h}_{F,t}$ over hedged returns $\mathbf{rx}_{F,t+1} = \mathbf{H}'_F \mathbf{rx}_{t+1}$. Thus, they solve

$$\max_{\mathbf{h}_{F,t}} \left\{ \mathbf{h}'_{F,t} E_t \left[\mathbf{rx}_{F,t+1}\right] - \frac{\tau}{2} \mathbf{h}'_{F,t} Var_t \left[\mathbf{rx}_{F,t+1}\right] \mathbf{h}_{F,t} \right\}.$$

As above, the solution is

$$\mathbf{h}_{F,t} = \tau \left(Var_t \left[\mathbf{rx}_{F,t+1}\right]\right)^{-1} E_t \left[\mathbf{rx}_{F,t+1}\right] = \tau \left(\mathbf{H}'_F Var_t \left[\mathbf{rx}_{t+1}\right] \mathbf{H}_F\right)^{-1} \mathbf{H}'_F E_t \left[\mathbf{rx}_{t+1}\right],$$

Following the same logic as above for domestic bond investors, we then have

$$\mathbf{d}_{F,t} = \tau \mathbf{H}_F \mathbf{V}^{-1} \mathbf{H}'_F E_t \left[\mathbf{rx}_{t+1}\right].$$

Unlike in our baseline model, we need not have $\mathbf{d}_{F,t} = \mathbf{d}_{D,t}$. This is because CIP deviations affect the hedged returns that investors earn in non-local long-term bonds.

Balance-sheet constrained banks The only players who can engage in the riskless CIP arbitrage are a set of balance-sheet constrained banks. These banks choose the value of their positions in the CIP arbitrage trade, $d^{cip}_{B,t}$, to solve

$$\max_{d^{cip}_{B,t}} \left\{ x^{cip}_t d^{cip}_{B,t} - (\kappa/2) \, (d^{cip}_{B,t})^2 \right\}, \tag{53}$$

where $\kappa \geq 0$ and $(\kappa/2)\,(d^{cip}_{B,t})^2$ captures non-risk-based balance sheet costs faced by banks. Thus, banks take a position in the CIP arbitrage trade equal to

$$d^{cip}_{B,t} = \kappa^{-1} x^{cip}_t,$$

or

$$\mathbf{d}_{B,t} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \kappa^{-1} \end{bmatrix} \begin{bmatrix} E_t \left[rx^{y}_{t+1}\right] \\ E_t [rx^{y^*}_{t+1}] \\ E_t [rx^{q}_{t+1}] \\ x^{cip}_t \end{bmatrix} = \mathbf{K} E_t \left[\mathbf{rx}_{t+1}\right].$$

Market clearing In this extension, we assume that s^q_t is exogenous net supply of risky FX exposure on a *forward basis* that bond investors must hold. And we assume that s^{cip}_t is the exogenous supply of the riskless CIP arbitrage trade that banks must undertake. We parameterize the exogenous supply shocks in this way to clearly separate the supply of risky FX exposure and the supply of riskless funding that bond investors and banks must intermediate. Thus, the exogenous supply of the four underlying long-short trades are

$$\mathbf{S}^{[3-7]}_1 \mathbf{s}_t = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} s^y_t \\ s^{y*}_t \\ s^q_t \\ s^{cip}_t \end{bmatrix}.$$

Below, we explain how the results change if s^q_t is exogenous net supply of risky FX exposure on a cash basis.

Here we assume that $s^{cip}_{t+1} = \phi_{s^{cip}} s^{cip}_t + \varepsilon_{s^{cip}_{t+1}}$, where $Var_t[\varepsilon_{s^{cip}_{t+1}}] = \sigma^2_{s^{cip}} \geq 0$, $\phi_{s^{cip}} \in [0, 1)$, and $\varepsilon_{s^{cip}_{t+1}}$ is orthogonal to the other shocks. The main text considers the special case where $\sigma^2_{s^{cip}} = 0$, implying that $s^{cip}_t \equiv 0$.

The market clearing conditions are

$$\mathbf{S}_1^{[3-7]}\mathbf{s}_t = \frac{1}{2}\left(\mathbf{d}_{D,t}+\mathbf{d}_{F,t}\right)+\mathbf{d}_{B,t} = \left[\frac{1}{2}\left(\tau\mathbf{H}_D\mathbf{V}^{-1}\mathbf{H}_D'+\tau\mathbf{H}_F\mathbf{V}^{-1}\mathbf{H}_F'\right)+\mathbf{K}\right]E_t\left[\mathbf{rx}_{t+1}\right],$$

implying that

$$E_t\left[\mathbf{rx}_{t+1}\right] = \left[\frac{1}{2}\left(\tau\mathbf{H}_D\mathbf{V}^{-1}\mathbf{H}_D'+\tau\mathbf{H}_F\mathbf{V}^{-1}\mathbf{H}_F'\right)+\mathbf{K}\right]^{-1}\mathbf{S}_1^{[3-6]}\mathbf{s}_t.$$

Working through the math, we obtain

$$\begin{bmatrix} E_t\left[rx_{t+1}^{y}\right] \\ E_t\left[rx_{t+1}^{y*}\right] \\ E_t\left[rx_{t+1}^{q}\right] \\ x_t^{cip} \end{bmatrix} = \left(\tau^{-1}\begin{bmatrix} V_y & C_{y,y*} & C_{y,q} & 0 \\ C_{y,y*} & V_y & -C_{y,q} & 0 \\ C_{y,q} & -C_{y,q} & V_q & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \kappa\frac{V_y+C_{y,y*}}{2V_y+2C_{y,y*}+\tau\kappa}\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & -1 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 1 \\ 1 & -1 & 0 & -2 \\ -1 & 1 & 0 & 2 \end{bmatrix}\right)\begin{bmatrix} s_t^{y} \\ s_t^{y*} \\ s_t^{q} \\ s_t^{cip} \end{bmatrix}.$$

These are the expressions given in Proposition 5.

As in our baseline model, the variance-covariance matrix of excess returns is an equilibrium object. Specifically, a rational expectations equilibrium of the extended model is a fixed point of an operator involving the "price-impact" coefficients which govern how the supplies $\mathbf{s}_t = [s_t^y, s_t^{y*}, s_t^q, s_t^{cip}]'$ impact y_t, y_t^*, q_t, and x_t^{cip}. One can also recast the equilibrium as a fixed point problem involving the equilibrium variance-convariance matrix. However, as in the baseline model in Section 3, if $0 \leq \rho < 1$, $\sigma_{s^y}^2 \geq 0$, $\sigma_{s^q}^2 \geq 0$, we must have $C_{y,q} > 0$ in any stable equilibrium. Furthermore, we must have $V_y + C_{y,y*} = V_y\left(1 + Corr\left[rx_{t+1}^y, rx_{t+1}^{y*}\right]\right) > 0$ and, thus, $\left[\kappa\left(V_y + C_{y,y*}\right)\right]/\left[2V_y + 2C_{y,y*} + \tau\kappa\right] > 0$ in any equilibrium. Thus, since

$$\begin{aligned} E_t\left[rx_{t+1}^{q}\right] &= \tau^{-1}[\overbrace{C_{y,q}}^{>0}\times(s_t^y - s_t^{y*}) + \overbrace{V_q}^{>0}\times s_t^q] - x_t^{cip}, \\ x_t^{cip} &= \underbrace{-\kappa\frac{V_y+C_{y,y*}}{2\left(V_y+C_{y,y*}\right)+\tau\kappa}}_{<0}\times[(s_t^y - s_t^{y*}) - 2\times s_t^{cip}], \end{aligned}$$

it follows that the three supply shocks s_t^y, s_t^{y*}, and s_t^{cip} push $E_t[rx_{t+1}^q]$ and x_t^{cip} in opposite directions. As a result, these three supply shock shocks push q_t and x_t^{cip} in the same direction.

Finally, since the model's stable equilibrium is continuous in the model's underlying parameters, it follows that if we take the limit where $\kappa \to 0$, then the extended model in Section 4 converges to the baseline model considered in Section 3. Specifically, as $\kappa \to 0$, CIP holds ($x_t^{cip} \to 0$).

Alternate assumption on FX supply If s_t^q is instead the exogenous net supply of risky FX exposure on a spot basis, then the exogenous supplies of the four underlying long-short trades are

$$\mathbf{S}_1^{[3-7]}\mathbf{s}_t = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} s_t^{y} \\ s_t^{y*} \\ s_t^{q} \\ s_t^{cip} \end{bmatrix}.$$

In this case, we still have

$$
\begin{aligned}
E_t\left[rx^y_{t+1}\right] &= \tau^{-1}\left[V_y \times s^y_t + C_{y,y*} \times s^{y*}_t + C_{y,q} \times s^q_t\right] - x^{cip}_t/2, \\
E_t\left[rx^{y*}_{t+1}\right] &= \tau^{-1}\left[C_{y,y*} \times s^y_t + V_y \times s^{y*}_t - C_{y,q} \times s^q_t\right] + x^{cip}_t/2, \\
E_t\left[rx^q_{t+1}\right] &= \tau^{-1}\left[C_{y,q} \times (s^y_t - s^{y*}_t) + V_q \times s^q_t\right] - x^{cip}_t,
\end{aligned}
$$

as above. However, we now have

$$
x^{cip}_t = \underbrace{-\kappa\frac{V_y + C_{y,y*}}{2\left(V_y + C_{y,y*}\right) + \tau\kappa}}_{<0} \times [(s^y_t - s^{y*}_t) - 2 \times (s^{cip}_t - s^q_t)].
$$

Now shocks to s^q_t also push $E_t[rx^q_{t+1}]$ and x^{cip}_t in opposite directions. The intuition is simple. To clear the spot foreign exchange market at time t, investors must be willing to exchange domestic currency for foreign currency at today's spot rate with no agreement to reverse this exchange at a later date. Since $rx^q_{t+1} = (q_{t+1} - f^q_t) - x^{cip}_t$, this spot investment in foreign currency is equivalent to a (i) forward investment in foreign currency plus (ii) a reverse FX swap—i.e., an exchange of domestic for foreign currency at today's spot rate $(1/Q_t)$ and simultaneous agreement to exchange foreign for domestic currency tomorrow at today's 1-period forward rate (F^Q_t). This reverse FX swap is equivalent to a reverse CIP arbitrage trade that borrows on a cash basis and lends on a synthetic basis in domestic currency.

To accommodate an inelastic demand s^q_t to swap foreign for domestic currency in the spot FX market, (i) risk-averse bond investors must make a forward investment in foreign currency in amount s^q_t and (ii) balance-sheet constrained banks must enter into a reverse CIP arbitrage trade, thereby earning $-x^{cip}_t$ in amount s^q_t. Thus, an increase in s^q_t will be associated with (i) an increase in the expected return to buying foreign currency on a forward basis, $E_t\left[q_{t+1} - f^q_t\right]$, and (ii) a decline in x^{cip}_t. The rise in $E_t\left[q_{t+1} - f^q_t\right]$ is required to induce risk-averse bond investors increase their risky forward investments in foreign currency. And, the decline in x^{cip}_t is required to induce balance-sheet constrained banks to supply reverse FX swaps. Combing these results, an increase in s^q_t must then lead $E_t\left[rx^q_{t+1}\right] = E_t\left[q_{t+1} - f^q_t\right] - x^{cip}_t$ to rise. Thus, shocks to s^q_t also push $E_t[rx^q_{t+1}]$ and x^{cip}_t in opposite directions.

C Model extensions

C.1 Further segmenting the global bond market

In the Section 5.1, we further segment the global bond market as in Gromb and Vayanos (2002), assuming some bond investors cannot trade short- and long-term bonds in both currencies. Our extended model feature four types of bond investors. All types have mean-variance preferences over one-period-ahead wealth and a risk tolerance of τ in domestic currency terms. The types only differ in their ability to trade different assets. Specifically, the four investor types are:

1. *Domestic bond specialists*, present in mass $\mu\pi$, can only choose between short- and long-term domestic bonds—i.e., they can only engage in the domestic yield-curve carry trade. Thus, their demand for long-term domestic bonds is $b^y_t = \tau\left(Var_t\left[rx^y_{t+1}\right]\right)^{-1} E_t\left[rx^y_{t+1}\right]$.

2. *Foreign bond specialists*, also present in mass $\mu\pi$, can only choose between short- and long-term foreign bonds—i.e., they can only engage in the foreign yield-curve carry trade. Their demand for long-term foreign bonds is $b^{y*}_t = \tau\left(Var_t\left[rx^{y*}_{t+1}\right]\right)^{-1} E_t\left[rx^{y*}_{t+1}\right]$.

3. *FX specialists*, present in mass $\mu(1-2\pi)$, can only choose between short-term domestic and foreign bonds—i.e., they can only engage in the FX carry trade. Their demand for the borrow-at-home-lend-abroad FX carry trade is $b_t^q = \tau\left(Var_t\left[rx_{t+1}^q\right]\right)^{-1} E_t\left[rx_{t+1}^q\right]$.

4. *Global bond investors*, present in mass $(1-\mu)$, can hold short- and long-term bonds in both currencies and can engage in all three carry trades. Their demand for the three carry trades is $\mathbf{d}_t = \tau\left(Var_t\left[\mathbf{rx}_{t+1}\right]\right)^{-1} E_t\left[\mathbf{rx}_{t+1}\right]$.

We assume $\mu \in [0,1]$ and $\pi \in (0,1/2)$. Thus, increasing the combined mass of specialist types, μ, is equivalent to introducing greater segmentation in the global bond market. Our baseline model corresponds to the limiting case where $\mu = 0$. At the other extreme, markets are fully segmented when $\mu = 1$. And, when $\mu \in (0,1)$ markets are partially segmented.

Technical assumption: Adding FX-specific fundamental risk To solve the extended model in the absence of supply risk, we assume there is some small amount of FX-specific fundamental risk. Naturally, this implies that long-run UIP will not hold even in the $\delta \to 1$ limit. We make this assumption to study our extended model in the absence of supply risk.

Specifically, we assume $\lim_{T\to\infty} E_t\left[q_{t+T}\right] = q_t^{\infty}$ follows a random walk $q_{t+1}^{\infty} = q_t^{\infty} + \varepsilon_{q^{\infty},t+1}$ with $Var_t\left[\varepsilon_{q^{\infty},t+1}\right] = \sigma_{q^{\infty}}^2 > 0$, implying $q_t = q_t^{\infty} + \sum_{j=0}^{\infty} E_t[(i_{t+j}^* - i_{t+j}) - rx_{t+j+1}^q]$. Thus, in the absence of supply risk, we have:

$$\mathbf{V} = \begin{bmatrix} \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \sigma_i^2 & \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \rho\sigma_i^2 & \frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2(1-\rho) \\ \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \rho\sigma_i^2 & \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \sigma_i^2 & -\frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2(1-\rho) \\ \frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2(1-\rho) & -\frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2(1-\rho) & \sigma_{\infty}^2 + \left(\frac{1}{1-\phi_i}\right)^2 2\sigma_i^2(1-\rho) \end{bmatrix}. \tag{56}$$

If $\sigma_{q^{\infty}}^2 = 0$, then in the absence of supply risk, FX is a redundant asset: FX returns are a linear combination of those on domestic and foreign bonds. While the model can still be solved in the limit where $\sigma_{q^{\infty}}^2 = \sigma_{s^q}^2 = \sigma_{s^y}^2 = 0$, in this case, cross-market impact increases in μ for all $\mu \in (0,1)$ and then discontinuously vanishes at $\mu = 1$. Thus, assuming $\sigma_{q^{\infty}}^2 > 0$ is a technical modeling device that allows us to explore the model's qualitative behavior when $\sigma_{s^q}^2, \sigma_{s^y}^2 > 0$ under the mathematically simpler assumption that $\sigma_{s^q}^2 = \sigma_{s^y}^2 = 0$. Specifically, when $\sigma_{s^q}^2$ and $\sigma_{s^y}^2$ are small and positive the model must have the same qualitative behavior as when $\sigma_{q^{\infty}}^2 > 0$.

Solution under further segmentation It is straightforward to solve for equilibrium under partial segmentation. Specifically, let $\mathbf{b}_t = [b_t^y, b_t^{y*}, b_t^q]'$ denote the vector of specialist investor demands and note that

$$\mathbf{b}_t = \tau\left[\mathbf{diag}\left(\mathbf{V}\right)\right]^{-1} E_t\left[\mathbf{rx}_{t+1}\right], \tag{57}$$

where $[\mathbf{diag}(\mathbf{V})]$ is a matrix with the diagonal elements of $\mathbf{V} = Var_t\left[\mathbf{rx}_{t+1}\right]$ on its diagonal and zeros elsewhere. Also let $\mathbf{\Pi} = \mathbf{diag}(\pi, \pi, 1-2\pi)$. The market clearing condition once we further segment the global rates market is

$$\begin{aligned} \mathbf{s}_t &= \mu\mathbf{\Pi}\mathbf{b}_t + (1-\mu)\,\mathbf{d}_t \\ &= \mu\mathbf{\Pi}\tau\left[\mathbf{diag}\left(\mathbf{V}\right)\right]^{-1} E_t\left[\mathbf{rx}_{t+1}\right] + (1-\mu)\,\tau\mathbf{V}^{-1}E_t\left[\mathbf{rx}_{t+1}\right]. \end{aligned} \tag{58}$$

As a result, equilibrium expected returns are:

$$E_t\left[\mathbf{rx}_{t+1}\right] = \tau^{-1}\left[\mu\mathbf{\Pi}\left[\mathbf{diag}\left(\mathbf{V}\right)\right]^{-1} + (1-\mu)\,\mathbf{V}^{-1}\right]^{-1}\mathbf{s}_t. \tag{59}$$

Thus, adopting the notation from above, the market clearing condition under further segmentation can be expressed as

$$\begin{aligned} & [\mathbf{B}_0\mathbf{a} + \mathbf{B}_1\mathbf{a} + \mathbf{r}_0] + [\mathbf{B}_0\mathbf{A} + \mathbf{B}_1\mathbf{A}\mathbf{\Phi} + \mathbf{R}_1]\,\mathbf{z}_t \qquad (60) \\ = \; & \tau^{-1}\left[\mu\mathbf{\Pi}\left[\mathbf{diag}\left(\mathbf{V}\right)\right]^{-1} + (1-\mu)\,\mathbf{V}^{-1}\right]^{-1}\mathbf{s}_0 + \tau^{-1}\left[\mu\mathbf{\Pi}\left[\mathbf{diag}\left(\mathbf{V}\right)\right]^{-1} + (1-\mu)\,\mathbf{V}^{-1}\right]^{-1}\mathbf{S_1}\mathbf{z}_t, \end{aligned}$$

where $\mathbf{V} = (\mathbf{B}_1\mathbf{A}\mathbf{\Sigma}\mathbf{A}'\mathbf{B}_1')$.

Matching the matrices multiplying the state vector $\mathbf{x}_t$, we see that $\mathbf{A}$ must solve the following fixed point problem:

$$\mathbf{A} = \left[\tau^{-1}\left[\mu\mathbf{\Pi}\left[\mathbf{diag}\left(\mathbf{B}_1\mathbf{A}\mathbf{\Sigma}\mathbf{A}'\mathbf{B}_1'\right)\right]^{-1} + (1-\mu)\left(\mathbf{B}_1\mathbf{A}\mathbf{\Sigma}\mathbf{A}'\mathbf{B}_1'\right)^{-1}\right]^{-1}\mathbf{S_1} - \mathbf{R}_1\right] \oslash [\mathbf{B}_0\mathbf{E} + \mathbf{B}_1\mathbf{E}\mathbf{\Phi}]. \qquad (61)$$

As above, partitioning $\mathbf{A}$ as $\mathbf{A} = [\mathbf{A}_i \;\; \mathbf{A}_s]$, we find that

$$\mathbf{A}_i = -\mathbf{R}_1^{[1-2]} \oslash [\mathbf{B}_0\mathbf{E} + \mathbf{B}_1\mathbf{E}\mathbf{\Phi}]^{[1-2]} = \begin{bmatrix} \frac{1-\delta}{1-\delta\phi_i} & 0 \\ 0 & \frac{1-\delta}{1-\delta\phi_i} \\ -\frac{1}{1-\phi_i} & \frac{1}{1-\phi_i} \end{bmatrix}.$$

Letting $\mathbf{V}_i = (\mathbf{B}_1\mathbf{A}_i)\,\mathbf{\Sigma}_i\,(\mathbf{B}_1\mathbf{A}_i)'$, we see that

$$\mathbf{V} = \mathbf{V}_i + (\mathbf{B}_1\mathbf{A}_s)\,\mathbf{\Sigma}_s\,(\mathbf{B}_1\mathbf{A}_s)'.$$

Thus, $\mathbf{A}_s$ must solve the following fixed point problem

$$\mathbf{A}_s = \left[\tau^{-1}\left[\begin{array}{c} \mu\mathbf{\Pi}\left(\mathbf{diag}\left[\mathbf{V}_i + (\mathbf{B}_1\mathbf{A}_s)\,\mathbf{\Sigma}_s\,(\mathbf{B}_1\mathbf{A}_s)'\right]\right)^{-1} \\ +(1-\mu)\left[\mathbf{V}_i + (\mathbf{B}_1\mathbf{A}_s)\,\mathbf{\Sigma}_s\,(\mathbf{B}_1\mathbf{A}_s)'\right]^{-1} \end{array}\right]^{-1}\right] \oslash [\mathbf{B}_0\mathbf{E} + \mathbf{B}_1\mathbf{E}\mathbf{\Phi}]^{[3-5]}. \qquad (62)$$

We also see that $\mathbf{a}$ must satisfy

$$(\mathbf{B}_0{+}\mathbf{B}_1)\,\mathbf{a} = \tau^{-1}\left[\mu\mathbf{\Pi}\left[\mathbf{diag}\left(\mathbf{B}_1\mathbf{A}\mathbf{\Sigma}\mathbf{A}'\mathbf{B}_1'\right)\right]^{-1} + (1-\mu)\left(\mathbf{B}_1\mathbf{A}\mathbf{\Sigma}\mathbf{A}'\mathbf{B}_1'\right)^{-1}\right]^{-1}\mathbf{s}_0 - \mathbf{r}_0. \qquad (63)$$

Recasting the equilibrium as a fixed point involving $\mathbf{\Omega}$ We can think of equilibrium as a fixed point problem involving the return impact matrix, $\mathbf{\Omega}$, that maps changes in asset to supply to shifts in asset returns—i.e. $E_t\left[\mathbf{rx}_{t+1}\right] = \mathbf{\Omega}\mathbf{s}_t$. This return impact matrix is given by

$$\mathbf{\Omega} = \tau^{-1}\left[\mu\mathbf{\Pi}\left[\mathbf{diag}\left(\mathbf{V}\right)\right]^{-1} + (1-\mu)\,\mathbf{V}^{-1}\right]^{-1}.$$

Thus, $\mathbf{\Omega}$ depends on the harmonic mean of $\mathbf{\Pi}^{-1}\left[\mathbf{diag}\left(\mathbf{V}\right)\right]$ and $\mathbf{V}$. Since $\mathbf{A}_s = \mathbf{\Omega} \oslash \mathbf{Z}_s$ where $\mathbf{Z}_s = [\mathbf{B}_0\mathbf{E} + \mathbf{B}_1\mathbf{E}\mathbf{\Phi}]^{[3-5]}$, we have

$$\mathbf{V} = \mathbf{V}_i + (\mathbf{B}_1\mathbf{\Omega} \oslash \mathbf{Z}_s)\,\mathbf{\Sigma}_s\,(\mathbf{B}_1\mathbf{\Omega}\,(\mu) \oslash \mathbf{Z}_s)' = \mathbf{V}_i + (\mathbf{\Omega} \circ \mathbf{Z})\,\mathbf{\Sigma}_s\,(\mathbf{\Omega} \circ \mathbf{Z})'$$

where $\mathbf{Z}$ satisfies $(\mathbf{B}_1\mathbf{\Omega} \oslash \mathbf{Z}_s) = \mathbf{\Omega} \circ \mathbf{Z}$. Thus, the relevant fixed point problem is

$$\mathbf{\Omega} = \tau^{-1}\left[\mu\mathbf{\Pi}\left[\mathbf{diag}\left(\mathbf{V}_i + (\mathbf{\Omega} \circ \mathbf{Z})\,\mathbf{\Sigma}_s\,(\mathbf{\Omega} \circ \mathbf{Z})'\right)\right]^{-1} + (1-\mu)\left[\mathbf{V}_i + \mathbf{Z}\circ(\mathbf{\Omega}\mathbf{\Sigma}_s\mathbf{\Omega}) \circ \mathbf{Z}'\right]^{-1}\right]^{-1}.$$

We use $\mathbf{\Omega}\,(\mu)$ to denote the stable solution to this fixed-point problem. We write $\mathbf{\Omega}\,(\mu)$ to emphasize that this solution depends on the degree of segmentation μ.

Understanding how $\mathbf{\Omega}(\mu)$ varies as a function of μ. We first want to understand how this solution $\mathbf{\Omega}(\mu)$ varies as a function of μ. Clearly, $\mathbf{\Omega}(\mu)$ is positive definite. However, we are interested in understanding when/whether $\mathbf{\Omega}'(\mu) = \partial \mathbf{\Omega}(\mu) / \partial \mu$ is itself a positive definite matrix.

Letting

$$\mathbf{\Omega}(\mu) = \tau^{-1} \left[\mu \mathbf{\Pi} \left[\mathbf{diag}\left(\mathbf{V}(\mu) \right) \right]^{-1} + (1-\mu) \mathbf{V}(\mu)^{-1} \right]^{-1}$$

and

$$\mathbf{V}(\mu) = \mathbf{V}_i + \left(\mathbf{\Omega}(\mu) \circ \mathbf{Z} \right) \mathbf{\Sigma}_s \left(\mathbf{\Omega}(\mu) \circ \mathbf{Z} \right)'$$

denote the equilibrium return-impact and variance matrices when fraction μ of investors are specialists. Using the rules of matrix differentiation, we obtain:

$$\begin{aligned} \mathbf{\Omega}'(\mu) &= \left\{ \tau \mathbf{\Omega}(\mu) \left[\mathbf{V}(\mu)^{-1} - \mathbf{\Pi} \left[\mathbf{diag}\left(\mathbf{V}(\mu) \right) \right]^{-1} \right] \mathbf{\Omega}(\mu) \right\} \\ &\quad + \left\{ \tau \mathbf{\Omega}(\mu) \left[\begin{array}{c} \mu \mathbf{\Pi} \left[\mathbf{diag}\left(\mathbf{V}(\mu) \right) \right]^{-1} \left[\mathbf{diag}\left(\mathbf{V}'(\mu) \right) \right] \left[\mathbf{diag}\left(\mathbf{V}(\mu) \right) \right]^{-1} \\ + (1-\mu) \mathbf{V}(\mu)^{-1} \mathbf{V}'(\mu) \mathbf{V}(\mu)^{-1} \end{array} \right] \mathbf{\Omega}(\mu) \right\} \end{aligned} \tag{64}$$

where

$$\mathbf{V}'(\mu) = \left(\mathbf{\Omega}'(\mu) \circ \mathbf{Z} \right) \mathbf{\Sigma}_s \left(\mathbf{\Omega}(\mu) \circ \mathbf{Z} \right)' + \left(\mathbf{\Omega}(\mu) \circ \mathbf{Z} \right) \mathbf{\Sigma}_s \left(\mathbf{\Omega}'(\mu) \circ \mathbf{Z} \right)'. \tag{65}$$

Below, we will show that $\mathbf{V}(\mu)^{-1} - \mathbf{\Pi} \left[\mathbf{diag}\left(\mathbf{V}(\mu) \right) \right]^{-1}$ is also positive definite, immediately implying that the first matrix in curly braces in equation (64) positive definite. This means that $\mathbf{\Omega}'(\mu)$ must always be positive definite in the absence of supply risk. Furthermore, by continuity of the stable equilibrium in the model's underlying parameters, $\mathbf{\Omega}'(\mu)$ must continue to be positive-definite when supply risk is small ($\mathbf{\Sigma}_s$ is small).

Furthermore, *if* $\mathbf{V}'(\mu)$ in equation (65) positive definite, then the second matrix in curly braces in equation (64) is also positive definite. Since the sum of positive definite matrices is also positive definite, this means that $\mathbf{\Omega}'(\mu)$ is also positive definite. (If $\mathbf{V}'(\mu)$ is positive definite, then we have $\partial Var_t[rx_{t+1}^{p_t}]/\partial \mu > 0$ for *any arbitrary* bond portfolio $\mathbf{p}_t \neq \mathbf{0}$ with returns $rx_{t+1}^{p_t} = \mathbf{p}_t' \mathbf{rx}_{t+1}$—i.e., return volatility is increasing in μ).

In our numerical solutions, we find $\partial Var_t[rx_{t+1}^{p_t}]/\partial \mu > 0$ for any portfolio $rx_{t+1}^{p_t} = \mathbf{p}_t' \mathbf{rx}_{t+1}$ so long as $\sigma_{s^y}^2$ and $\sigma_{s^q}^2$ have a similar order of magnitude. However, $\partial Var_t[rx_{t+1}^{p_t}]/\partial \mu$ can be negative when $\sigma_{s^y}^2$ and $\sigma_{s^q}^2$ have different orders of magnitude. For instance, suppose $\sigma_{s^q}^2 > 0$ and $\sigma_{s^y}^2 = 0$. Then $Var_t[rx_{t+1}^y | \mu = 0] > Var_t[rx_{t+1}^y | \mu = 1]$ since FX supply shocks raise the volatility of bond returns in integrated markets ($\mu = 0$) but not in fully segmented markets ($\mu = 1$). As a result, there exists some $\mu^* \in [0, 1]$ such that $\partial Var_t[rx_{t+1}^y | \mu = \mu^*]/\partial \mu < 0$.

Proof that $\mathbf{\Omega}'(\mu)$ is positive definite in the absence of supply risk. We now show that $\left[\mathbf{V}^{-1} - \mathbf{\Pi} \left[\mathbf{diag}(\mathbf{V}) \right]^{-1} \right]$ and therefore $\mathbf{\Omega}'(\mu) = \left\{ \tau \mathbf{\Omega}(\mu) \left[\mathbf{V}^{-1} - \mathbf{\Pi} \left[\mathbf{diag}(\mathbf{V}) \right]^{-1} \right] \mathbf{\Omega}(\mu) \right\}$ is positive definite in the absence of supply risk ($\mathbf{\Sigma}_s = \mathbf{0}$). This is actually true for any positive-definite covariance matrix $\mathbf{V}$ and any diagonal, positive-definite matrix $\mathbf{\Pi}$ such that $trace(\mathbf{\Pi}) = 1$.

Here will simply prove this for the special case that is relevant for us in the paper. We write $\mathbf{V} = \mathbf{S\Gamma S}$ where $\mathbf{S}$ is a diagonal matrix with the standard deviation of excess returns on its diagonals and $\mathbf{\Gamma}$ is the correlation matrix for the excess returns. Since

$$\mathbf{V}^{-1} - \mathbf{\Pi} \left[\mathbf{diag}(\mathbf{V}) \right]^{-1} = \mathbf{S}^{-1} (\mathbf{\Gamma}^{-1} - \mathbf{\Pi}) \mathbf{S}^{-1}$$

it suffices to show that $\mathbf{\Gamma}^{-1}-\mathbf{\Pi}$ is positive definite. We have

$$\mathbf{\Gamma}^{-1}-\mathbf{\Pi}=\left(\begin{bmatrix}1 & \gamma_y & \gamma_q\\ \gamma_y & 1 & -\gamma_q\\ \gamma_q & -\gamma_q & 1\end{bmatrix}^{-1}-\begin{bmatrix}\pi & 0 & 0\\ 0 & \pi & 0\\ 0 & 0 & 1-2\pi\end{bmatrix}\right)$$

where $\pi\in(0,1/2)$ and $\gamma_y\in(-1,1)$ and $\gamma_q\in(-1,1)$.

We begin by noting that, since $\mathbf{\Gamma}$ is positive definite, the eigenvalues of $\mathbf{\Gamma}$ are positive. The eigenvalues of $\mathbf{\Gamma}$ are $1+\gamma_y$, $1-\frac{1}{2}\gamma_y+\frac{1}{2}\sqrt{8\gamma_q^2+\gamma_y^2}$, $1-\frac{1}{2}\gamma_y-\frac{1}{2}\sqrt{8\gamma_q^2+\gamma_y^2}$. Thus, we have $2-\gamma_y>\sqrt{8\gamma_q^2+\gamma_y^2}>-\left(2-\gamma_y\right)$. The fact that $1-\frac{1}{2}\gamma_y-\frac{1}{2}\sqrt{8\gamma_q^2+\gamma_y^2}>0$ also implies that $1-\gamma_y-2\gamma_q^2>0$. We will use this fact repeatedly below.

Next, the three eigenvalues of $\mathbf{\Gamma}^{-1}-\mathbf{\Pi}$ are:

1. $\lambda_1=\frac{1}{2}\frac{[1+\pi\left(1-\gamma_y\right)+(1-\pi)2\gamma_q^2]+\sqrt{[1+\pi\left(1-\gamma_y\right)+(1-\pi)2\gamma_q^2]^2-8\pi(1-\gamma_y-2\gamma_q^2)(1-\pi\left(1-\gamma_y\right)-\gamma_q^2(1-2\pi))}}{1-\gamma_y-2\gamma_q^2}>0.$

   - As noted above, we have $1-\gamma_y-2\gamma_q^2>0$.
   - We also have $1+\pi\left(1-\gamma_y\right)+(1-\pi)\,2\gamma_q^2>0$.
   - Since $\mathbf{\Gamma}^{-1}-\mathbf{\Pi}$ is symmetric, the eigenvalues of $\mathbf{\Gamma}^{-1}-\mathbf{\Pi}$ are real-valued. Thus, the term under the radical is positive.
   - Together, these three facts imply that $\lambda_1>0$.

2. $\lambda_2=\frac{1}{2}\frac{[1+\pi\left(1-\gamma_y\right)+(1-\pi)2\gamma_q^2]-\sqrt{[1+\pi\left(1-\gamma_y\right)+(1-\pi)2\gamma_q^2]^2-8\pi(1-\gamma_y-2\gamma_q^2)(1-\pi\left(1-\gamma_y\right)-\gamma_q^2(1-2\pi))}}{1-\gamma_y-2\gamma_q^2}>0.$

   - In addition to the facts listed above, this follows from the fact that

   $$8\pi\left(1-\gamma_y-2\gamma_q^2\right)\left(1-\pi\left(1-\gamma_y\right)-\gamma_q^2\,(1-2\pi)\right)>0.$$

   Specifically, $\left(1-\gamma_y-2\gamma_q^2\right)>0$ and, since $\gamma_y\in(-1,1)$, $\left(1-\pi\left(1-\gamma_y\right)-\gamma_q^2\,(1-2\pi)\right)>(1-2\pi)\left(1-\gamma_q^2\right)>0$. Finally, since for $X>0$, $Y>0$, and $X^2-Y>0$ together imply $X-\sqrt{X^2-Y}>0$, we conclude that $\lambda_2>0$.

3. $\lambda_3=\frac{1}{1+\gamma_y}\left(1-\pi-\pi\gamma_y\right)>0.$

   - This follows from the fact that $1-2\pi>0$ and $\gamma_y\in(-1,1)$ which together imply that $1-\pi-\pi\gamma_y>1-2\pi>0$.

Interpreting $\mathbf{\Omega}'(\mu)$. To interpret $\mathbf{\Omega}'(\mu)$ in equation (64), we note that increasing μ—i.e., further segmenting the global rates markets—has two direct equilibrium effects. First, as we increase μ, risk sharing becomes less efficient because fewer investors can absorb a given supply shock. For instance, the fraction of investors who can absorb a shock to domestic bond supply is $\mu\pi+(1-\mu)$, which is decreasing in μ. This gives rise to the "inefficient risk-sharing" effect. Second, as we increase μ, we replace global bond investors whose demands take the correlations between the three carry trades into account with specialist investors who, taken as a group, behave as if the three carry trade returns are uncorrelated. This gives rise to the "width of the pipe" effect: price impact is only transmitted across markets to the extent there are investors—"the pipe"—whose demands are impacted by shocks to other markets. Finally, there is a third indirect effect of increasing segmentation. To the extent that greater segmentation directly alters the price impact of supply shocks, greater segmentation affects equilibrium return volatility, further altering equilibrium price impact. This is an "endogenous risk effect."

Thus, we further decompose $\mathbf{\Omega}'(\mu)$ into three terms

$$\mathbf{\Omega}'(\mu) = \overbrace{\left\{\tau\mathbf{\Omega}(\mu)\left[[\mathbf{diag}(\mathbf{V}(\mu))]^{-1} - \mathbf{\Pi}[\mathbf{diag}(\mathbf{V}(\mu))]^{-1}\right]\mathbf{\Omega}(\mu)\right\}}^{\mathbf{\Omega}'_{\text{sharing}}(\mu) = \text{Inefficient risk sharing effect}} + \overbrace{\left\{\tau\mathbf{\Omega}(\mu)\left[\mathbf{V}(\mu)^{-1} - [\mathbf{diag}(\mathbf{V}(\mu))]^{-1}\right]\mathbf{\Omega}(\mu)\right\}}^{\mathbf{\Omega}'_{\text{pipe}}(\mu) = \text{Width of pipe effect}} + \overbrace{\left\{\tau\mathbf{\Omega}(\mu)\left[\begin{array}{c}\mu\mathbf{\Pi}[\mathbf{diag}(\mathbf{V}(\mu))]^{-1}[\mathbf{diag}(\mathbf{V}'(\mu))][\mathbf{diag}(\mathbf{V}(\mu))]^{-1} \\ +(1-\mu)\mathbf{V}(\mu)^{-1}\mathbf{V}'(\mu)\mathbf{V}(\mu)^{-1}\end{array}\right]\mathbf{\Omega}(\mu)\right\}}^{\mathbf{\Omega}'_{\text{risk}}(\mu) = \text{Endogenous risk effect}}. \quad (66)$$

First, if the mass of investors who could buy each asset were independent of μ—i.e., if we instead had $\mathbf{\Pi} = \mathbf{I}$, we would have $\mathbf{\Omega}'_{\text{sharing}}(\mu) = \mathbf{0}$. Second, if assets returns were uncorrelated—i.e., if we instead had $\mathbf{V}(\mu) = [\mathbf{diag}(\mathbf{V}(\mu))]$, we would have $\mathbf{\Omega}'_{\text{pipe}}(\mu) = \mathbf{0}$. Finally, if there was no supply risk—i.e., if $\mathbf{V}'(\mu) = \mathbf{0}$, then would have $\mathbf{\Omega}'_{\text{risk}}(\mu) = \mathbf{0}$.

Letting $\boldsymbol{\omega}^* = vec(\mathbf{\Omega}^*)$, we can this of our equilibrium as a fixed-point problem in $\boldsymbol{\omega}$

$$\boldsymbol{\omega}^* = \mathbf{f}(\boldsymbol{\omega}^*, \mu).$$

Thus, by the Implicit Function Theorem, we have

$$\frac{\partial\boldsymbol{\omega}^*}{\partial\mu} = [\mathbf{I} - \mathbf{D}_{\boldsymbol{\omega}}\mathbf{f}(\boldsymbol{\omega}^*, \mu)]^{-1}[\partial\mathbf{f}(\boldsymbol{\omega}^*, \mu)/\partial\mu],$$

where $\mathbf{D}_{\boldsymbol{\omega}}\mathbf{f}(\boldsymbol{\omega}^*, \mu)$ is the Jacobian matrix. Assuming we are at a stable equilibrium, we have $[\mathbf{I} - \mathbf{D}_{\boldsymbol{\omega}}\mathbf{f}(\boldsymbol{\omega}^*, \mu)]^{-1} = \sum_{j=0}^{\infty}[\mathbf{D}_{\boldsymbol{\omega}}\mathbf{f}(\boldsymbol{\omega}^*, \mu)]^j$, so we have

$$\begin{aligned}\partial\boldsymbol{\omega}^*/\partial\mu &= \textstyle\sum_{j=0}^{\infty}[\mathbf{D}_{\boldsymbol{\omega}}\mathbf{f}(\boldsymbol{\omega}^*, \mu)]^j[\partial\mathbf{f}(\boldsymbol{\omega}^*, \mu)/\partial\mu] \\ &= [\partial\mathbf{f}(\boldsymbol{\omega}^*, \mu)/\partial\mu] + [\mathbf{I} - \mathbf{D}_{\boldsymbol{\omega}}\mathbf{f}(\boldsymbol{\omega}^*, \mu)]^{-1}\mathbf{D}_{\boldsymbol{\omega}}\mathbf{f}(\boldsymbol{\omega}^*, \mu)[\partial\mathbf{f}(\boldsymbol{\omega}^*, \mu)/\partial\mu]\end{aligned}$$

where

$$\partial\mathbf{f}(\boldsymbol{\omega}^*, \mu)/\partial\mu = vec\left(\tau\mathbf{\Omega}(\mu)\left[\mathbf{V}(\mu)^{-1} - \mathbf{\Pi}[\mathbf{diag}(\mathbf{V}(\mu))]^{-1}\right]\mathbf{\Omega}(\mu)\right) = vec\left(\mathbf{\Omega}'_{\text{sharing}}(\mu) + \mathbf{\Omega}'_{\text{pipe}}(\mu)\right).$$

In the absence of supply risk ($\mathbf{\Sigma}_s = \mathbf{0}$), $\mathbf{D}_{\boldsymbol{\omega}}\mathbf{f}(\boldsymbol{\omega}^*, \mu) = \mathbf{0}$ and $\partial\boldsymbol{\omega}^*/\partial\mu = vec\left(\mathbf{\Omega}'_{\text{sharing}}(\mu) + \mathbf{\Omega}'_{\text{pipe}}(\mu)\right)$. In the presence of supply risk, $\mathbf{D}_{\boldsymbol{\omega}}\mathbf{f}(\boldsymbol{\omega}^*, \mu) \neq \mathbf{0}$ and

$$\partial\boldsymbol{\omega}^*/\partial\mu = \overbrace{vec[\mathbf{\Omega}'_{\text{sharing}}(\mu) + \mathbf{\Omega}'_{\text{pipe}}(\mu)]}^{\text{Direct effects}} + \overbrace{[\mathbf{I} - \mathbf{D}_{\boldsymbol{\omega}}\mathbf{f}(\boldsymbol{\omega}^*, \mu)]^{-1}\mathbf{D}_{\boldsymbol{\omega}}\mathbf{f}(\boldsymbol{\omega}^*, \mu)\,vec[\mathbf{\Omega}'_{\text{sharing}}(\mu) + \mathbf{\Omega}'_{\text{pipe}}(\mu)]}^{\text{Endogenous risk effect: Amplifies direct effects}}.$$

Thus, there is a clear mathematical sense in which the endogenous risk effect amplifies the two direct effects of a change in μ.

How individual elements of $\mathbf{\Omega}(\mu)$ behave as a function of μ.

Diagonal elements of $\mathbf{\Omega}(\mu)$ and $\mathbf{\Omega}'(\mu)$. Recall that the diagonal elements of a positive-definite matrix are positive. Then the facts that $\partial E_t[rx^a_{t+1}]/\partial s^a_t = \Omega_{aa} > 0$ and $\partial^2 E_t[rx^a_{t+1}]/\partial s^a_t\partial\mu =$

$\partial\Omega_{aa}/\partial\mu > 0$ for any $a \in \{y, y^*, q\}$ when supply risk is small follow immediately from the facts that both $\mathbf{\Omega}(\mu)$ and $\mathbf{\Omega}'(\mu)$ are positive-definite.

Off-diagonal elements of $\mathbf{\Omega}(\mu)$ and $\mathbf{\Omega}'(\mu)$. Again, we consider the limit with zero supply risk. We show that for any $a \in \{y, y^*, q\}$ and $a' \neq a$, $|\Omega_{aa'}(\mu)| = |\partial E_t[rx^a_{t+1}]/\partial s^{a'}_t|$ is hump-shaped function of μ that satisfies $|\Omega_{aa'}(0)| > 0$ and $\Omega_{aa'}(1) = 0$.

The shape of $|\Omega_{aa'}(\mu)|$ reflects the juxtaposition of an "inefficient risk-sharing" effect which is *typically* increasing in μ and a "width of the pipe" effect which is *typically* decreasing in μ. In certain special cases, we have shown that the "inefficient risk-sharing" effect is *always* increasing in μ and a "width of the pipe" effect which is *always* decreasing in μ. For instance, these results hold in a model like ours if (i) there are an equal number of specialists in each asset and (ii) the returns on all assets have the same correlation with each other. However, while these results typically hold for a randomly chosen correlation matrix and a $\mathbf{\Pi}$ matrix, counterexamples are possibility.

We assume that $\mathbf{V}$ is fixed—i.e., we ignore the endogenous risk effect—and adopt the notation that $\mathbf{V} = \mathbf{S\Gamma S}$ where $\mathbf{S}$ is a diagonal matrix of standard deviations and $\mathbf{\Gamma} = \mathbf{S}^{-1}\mathbf{V}\mathbf{S}^{-1}$ is the correlation matrix. Using this notation, we have

$$\mathbf{\Omega}(\mu) = \tau^{-1}\mathbf{S}\left[\mu\mathbf{\Pi} + (1-\mu)\mathbf{\Gamma}^{-1}\right]^{-1}\mathbf{S},$$

and

$$\begin{aligned}
\mathbf{\Omega}'(\mu) &= \tau^{-1}\mathbf{S}\left[\mu\mathbf{\Pi} + (1-\mu)\mathbf{\Gamma}^{-1}\right]^{-1}\overbrace{\left[\mathbf{\Gamma}^{-1} - \mathbf{\Pi}\right]}^{\text{Positive-definite}}\left[\mu\mathbf{\Pi} + (1-\mu)\mathbf{\Gamma}^{-1}\right]^{-1}\mathbf{S} \\
\mathbf{\Omega}'_{\text{sharing}}(\mu) &= \tau^{-1}\mathbf{S}\left[\mu\mathbf{\Pi} + (1-\mu)\mathbf{\Gamma}^{-1}\right]^{-1}\overbrace{\left[\mathbf{I} - \mathbf{\Pi}\right]}^{\text{Positive-definite}}\left[\mu\mathbf{\Pi} + (1-\mu)\mathbf{\Gamma}^{-1}\right]^{-1}\mathbf{S} \\
\mathbf{\Omega}'_{\text{pipe}}(\mu) &= \tau^{-1}\mathbf{S}\left[\mu\mathbf{\Pi} + (1-\mu)\mathbf{\Gamma}^{-1}\right]^{-1}\overbrace{\left[\mathbf{\Gamma}^{-1} - \mathbf{I}\right]}^{\text{Indefinite}}\left[\mu\mathbf{\Pi} + (1-\mu)\mathbf{\Gamma}^{-1}\right]^{-1}\mathbf{S}
\end{aligned}$$

Limit when $\mu = 0$. When $\mu = 0$, we have

$$\mathbf{\Omega}(0) = \tau^{-1}\mathbf{S\Gamma S} \text{ and } \mathbf{\Omega}'(0) = \tau^{-1}\mathbf{S\Gamma}\left[\mathbf{\Gamma}^{-1} - \mathbf{\Pi}\right]\mathbf{\Gamma S}.$$

Specifically, we have

$$\mathbf{\Omega}'(0) = \tau^{-1}\mathbf{S}\begin{bmatrix} \pi_2(1-\gamma_{12}^2) + \pi_3(1-\gamma_{13}^2) & \pi_3(\gamma_{12} - \gamma_{13}\gamma_{23}) & \pi_2(\gamma_{13} - \gamma_{12}\gamma_{23}) \\ \pi_3(\gamma_{12} - \gamma_{13}\gamma_{23}) & \pi_1(1-\gamma_{12}^2) + \pi_3(1-\gamma_{23}^2) & \pi_1(\gamma_{23} - \gamma_{12}\gamma_{13}) \\ \pi_2(\gamma_{13} - \gamma_{12}\gamma_{23}) & \pi_1(\gamma_{23} - \gamma_{12}\gamma_{13}) & \pi_1(1-\gamma_{13}^2) + \pi_2(1-\gamma_{23}^2) \end{bmatrix}\mathbf{S}.$$

Thus, the off-diagonals of $\mathbf{\Omega}(0)$ have the same signs as the corresponding univariate correlations. And, the diagonals of $\mathbf{\Omega}'(0)$ are always positive and off-diagonals have same signs as the corresponding *partia*l correlations.

Limit when $\mu = 1$. We have

$$\mathbf{\Omega}(1) = \tau^{-1}\mathbf{S\Pi}^{-1}\mathbf{S} \text{ and } \mathbf{\Omega}'(1) = \tau^{-1}(\mathbf{S\Pi}^{-1})[\mathbf{\Gamma}^{-1} - \mathbf{\Pi}](\mathbf{\Pi}^{-1}\mathbf{S}).$$

Using the properties of the inverse correlation matrix, we can show that the diagonals of $\mathbf{\Omega}'(1)$ are always positive and the off-diagonals have opposite signs as partial correlations. Specifically, we have the following results:

- *Diagonals of* $[\mathbf{\Gamma}^{-1}-\mathbf{\Pi}]$ *are positive*: The i^{th} diagonal element of $\mathbf{\Gamma}^{-1}$ is $[\mathbf{\Gamma}^{-1}]_{ii} = (1-R^2_{[\text{reg } i \text{ on all } i'\neq i]})^{-1} \geq 1$, where $R^2_{[\text{reg } i \text{ on all } i'\neq i]}$ is the R^2 from a regression of the ith variable on all other variables. Since $\mathbf{\Pi}$ is a diagonal matrix with diagonal elements strictly less 1 and the diagonal elements of $\mathbf{\Gamma}^{-1}$ greater than or equal to 1, the diagonal elements of $[\mathbf{\Gamma}^{-1} - \mathbf{\Pi}]$ are positive.

- *Off-diagonal elements of* $[\mathbf{\Gamma}^{-1} - \mathbf{\Pi}]$ *have signs opposite those of partial correlations*: The off-diagonals of $\mathbf{\Pi}$ are zero, so the off-diagonals of $[\mathbf{\Gamma}^{-1} - \mathbf{\Pi}]$ are just the off-diagonals of $\mathbf{\Gamma}^{-1}$. The off-diagonals are $\mathbf{\Gamma}^{-1}$ are $[\mathbf{\Gamma}^{-1}]_{ij} = -[\mathbf{\Gamma}^{-1}]_{ii} \times b_j^{[i]}$ where $b_j^{[i]}$ is the regression coefficient on the standardized version on variable j from a multivariate regression of standardized variable i on standardized versions all other variables. In other words, we have $(X_i - E[X_i])/\sigma[X_i] = \sum_{j\neq i} b_j^{[i]} \times (X_j - E[X_j])/\sigma[X_j] + \epsilon_i$.These $b_j^{[i]}$ are related to the coefficients from the more familiar multivariate regression $X_i = \sum_{j\neq i} \beta_j^{[i]} \times X_j + \varepsilon_i$ via $\beta_j^{[i]} = (\sigma[X_i]/\sigma[X_j]) \times b_j^{[i]}$. Thus, we have $[\mathbf{\Gamma}^{-1}]_{ij} = -\beta_j^{[i]} (\sigma[X_j]/\sigma[X_i]) [\mathbf{\Gamma}^{-1}]_{ii}$so $[\mathbf{\Gamma}^{-1}]_{ij}$ has the opposite sign as $\beta_j^{[i]}$.

For instance, for a 3×3 correlation matrix, we have

$$\begin{bmatrix} 1 & \gamma_{12} & \gamma_{13} \\ \gamma_{12} & 1 & \gamma_{23} \\ \gamma_{13} & \gamma_{23} & 1 \end{bmatrix}^{-1} = \frac{\begin{bmatrix} 1-\gamma_{23}^2 & -(\gamma_{12}-\gamma_{13}\gamma_{23}) & -(\gamma_{13}-\gamma_{12}\gamma_{23}) \\ -(\gamma_{12}-\gamma_{13}\gamma_{23}) & 1-\gamma_{13}^2 & -(\gamma_{23}-\gamma_{12}\gamma_{13}) \\ -(\gamma_{13}-\gamma_{12}\gamma_{23}) & -(\gamma_{23}-\gamma_{12}\gamma_{13}) & 1-\gamma_{12}^2 \end{bmatrix}}{1-\gamma_{12}^2-\gamma_{13}^2-\gamma_{23}^2+2\gamma_{12}\gamma_{13}\gamma_{23}},$$

where $\det(\mathbf{\Gamma}) = 1-\gamma_{12}^2-\gamma_{13}^2-\gamma_{23}^2+2\gamma_{12}\gamma_{13}\gamma_{23} > 0$. So we have

$$[\mathbf{\Gamma}^{-1}]_{11} = \frac{1}{1-\frac{\gamma_{12}^2+\gamma_{13}^2-2\gamma_{12}\gamma_{13}\gamma_{23}}{1-\gamma_{23}^2}}, \; b_2^{[1]} = \frac{\gamma_{12}-\gamma_{13}\gamma_{23}}{1-\gamma_{23}^2}, \text{ and } b_3^{[1]} = \frac{\gamma_{13}-\gamma_{12}\gamma_{23}}{1-\gamma_{23}^2}.$$

Global behavior of off-diagonal elements on $\mu \in$ Using these facts, we can then characterize the global behavior of the off-diagonal elements of $\mathbf{\Omega}(\mu)$

- We always have $\Omega_{ij}(1) = 0$.

- If $sign(\gamma_{ij}) = sign(\gamma_{ij}^P)$, then $sign(\Omega_{ij}(\mu)) = sign(\gamma_{ij})$ for all $\mu \in [0,1)$ and $\Omega_{ij}(\mu)$ is a hump-shaped function of μ.

  - If $\gamma_{ij} > 0$, then $\Omega_{ij}(\mu) > 0$ for all $\mu \in [0,1)$, $\partial\Omega_{ij}(\mu)/\partial\mu > 0$ for μ near 0, and $\partial\Omega_{ij}(\mu)/\partial\mu < 0$ for μ near 1—i.e., $\Omega_{ij}(\mu)$ is inverse U-shaped.
  - If $\gamma_{ij} < 0$, then $\Omega_{ij}(\mu) < 0$ for all $\mu \in [0,1)$, $\partial\Omega_{ij}(\mu)/\partial\mu < 0$ for μ near 0, and $\partial\Omega_{ij}(\mu)/\partial\mu > 0$ for μ near 1—i.e., a $\Omega_{ij}(\mu)$ is U-shaped.

- If $sign(\gamma_{ij}) \neq sign(\gamma_{ij}^P)$, then $sign(\Omega_{ij}(\mu)) = sign(\gamma_{ij})$ for μ near 0, $sign(\Omega_{ij}(\mu)) = sign(\gamma_{ij}^P)$ for μ near 1, and $\Omega_{ij}(\mu)$ is again a hump-shaped function of μ.

  - If $\gamma_{ij} > 0$ and $\gamma_{ij}^P < 0$, $\Omega_{ij}(\mu) > 0$ for μ near 0, $\Omega_{ij}(\mu) < 0$ for μ near 1, $\partial\Omega_{ij}(\mu)/\partial\mu < 0$ for μ near 0, and $\partial\Omega_{ij}(\mu)/\partial\mu > 0$ for μ near 1—i.e., $\Omega_{ij}(\mu)$ is U-shaped.
  - If $\gamma_{ij} < 0$ and $\gamma_{ij}^P > 0$, then $\Omega_{ij}(\mu) < 0$ for μ near 0, $\Omega_{ij}(\mu) > 0$ for μ near 1, $\partial\Omega_{ij}(\mu)/\partial\mu > 0$ for μ near 0, and $\partial\Omega_{ij}(\mu)/\partial\mu < 0$ for μ near 1—i.e., $\Omega_{ij}(\mu)$ is inverse U-shaped.

- If $\gamma_{ij}^P = 0$, then $sign(\Omega_{ij}(\mu)) = sign(\gamma_{ij})$ for all $\mu \in [0,1)$ and $|\Omega_{ij}(\mu)|$ is a monotonically decreasing function of μ.

Checking that univariate correlations equal partial correlations in our model when there is no supply risk. We now check that $sign\left(\gamma_{ij}\right) = sign(\gamma_{ij}^P)$ in our model when there is no supply risk. In the absence of supply risk, we have:

$$\mathbf{V} = \begin{bmatrix} \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \sigma_i^2 & \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \rho\sigma_i^2 & \frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2\left(1-\rho\right) \\ \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \rho\sigma_i^2 & \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \sigma_i^2 & -\frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2\left(1-\rho\right) \\ \frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2\left(1-\rho\right) & -\frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2\left(1-\rho\right) & \sigma_{q\infty}^2 + \left(\frac{1}{1-\phi_i}\right)^2 2\sigma_i^2\left(1-\rho\right) \end{bmatrix}.$$

- **Domestic bonds:** The partial correlation of domestic bonds with foreign bonds and FX are proportional to the regression coefficients:

$$\begin{aligned} &\begin{bmatrix} \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \sigma_i^2 & -\frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2\left(1-\rho\right) \\ -\frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2\left(1-\rho\right) & \sigma_{q\infty}^2 + \left(\frac{1}{1-\phi_i}\right)^2 2\sigma_i^2\left(1-\rho\right) \end{bmatrix}^{-1} \begin{bmatrix} \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \rho\sigma_i^2 \\ \frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2\left(1-\rho\right) \end{bmatrix} \\ = &\begin{bmatrix} 1 \\ \delta\frac{1-\phi_i}{1-\delta\phi_i} \end{bmatrix} - \frac{\left(1-\phi_i\right)^2}{\frac{\sigma_i^2}{\sigma_{q\infty}^2}\left(1-\rho^2\right) + \left(1-\phi_i\right)^2} \begin{bmatrix} \left(1-\rho\right) \\ \delta\frac{\left(1-\phi_i\right)}{\left(1-\delta\phi_i\right)} \end{bmatrix}. \end{aligned}$$

  These have the same signs as the univariate covariances:

$$\begin{bmatrix} \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \rho\sigma_i^2 \\ \frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2\left(1-\rho\right) \end{bmatrix}.$$

- **Foreign bonds:** The partial correlation of foreign bonds with domestic bonds and FX are proportional to the regression coefficients:

$$\begin{aligned} &\begin{bmatrix} \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \sigma_i^2 & \frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2\left(1-\rho\right) \\ \frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2\left(1-\rho\right) & \sigma_{q\infty}^2 + \left(\frac{1}{1-\phi_i}\right)^2 2\sigma_i^2\left(1-\rho\right) \end{bmatrix}^{-1} \begin{bmatrix} \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \rho\sigma_i^2 \\ -\frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2\left(1-\rho\right) \end{bmatrix} \\ = &\begin{bmatrix} 1 \\ -\delta\frac{1-\phi_i}{1-\delta\phi_i} \end{bmatrix} - \frac{\left(1-\phi_i\right)^2}{\frac{\sigma_i^2}{\sigma_{q\infty}^2}\left(1-\rho^2\right) + \left(1-\phi_i\right)^2} \begin{bmatrix} \left(1-\rho\right) \\ -\delta\frac{\left(1-\phi_i\right)}{\left(1-\delta\phi_i\right)} \end{bmatrix}. \end{aligned}$$

  These have the same signs as the univariate covariances:

$$\begin{bmatrix} \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \rho\sigma_i^2 \\ -\frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2\left(1-\rho\right) \end{bmatrix}.$$

- **Foreign exchange:** The partial correlation of foreign exchange with domestic and foreign bonds are proportional to the regression coefficients:

$$\begin{bmatrix} \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \sigma_i^2 & \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \rho\sigma_i^2 \\ \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \rho\sigma_i^2 & \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \sigma_i^2 \end{bmatrix}^{-1} \begin{bmatrix} \frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2\left(1-\rho\right) \\ -\frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2\left(1-\rho\right) \end{bmatrix} = \begin{bmatrix} \frac{(1-\delta\phi_i)}{\delta(1-\phi_i)} \\ -\frac{(1-\delta\phi_i)}{\delta(1-\phi_i)} \end{bmatrix}.$$

These have the same signs as the univariate covariances:

$$\begin{bmatrix} \frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2(1-\rho) \\ -\frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2(1-\rho) \end{bmatrix}.$$

Intuition for why cross-market price impact is hump-shaped. In fully integrated markets, price impact is determined by univariate correlations (univariate regression coefficients). In partially segmented markets, both univariate correlations and partial correlations (multivariate regression coefficients) matter. And, of course, the distinction between univariate correlations and partial correlations only arises once there are $N > 2$ assets.

When $\mu \to 1$, impact of a long-term domestic supply shock on long-term domestic returns is very large. Because of the strong hedging opportunities afforded by the opportunities to hedge in multiple asset classes, the small number of generalists trade quite aggressively, taking large short positions in foreign bonds and FX. These large positions have a large impact on prices in these markets. As markets become more integrated (μ falls), the impact on long-term domestic returns falls and generalists take smaller short positions, leading impact on foreign bonds and FX to decline.

As we increase μ, fewer investors can absorb a given supply shock. This inefficient risk sharing effect means that supply shocks have a larger impact on returns. At the same time, as we raise μ the width of the pipe effect means that price impact is more localized in the specific asset class where the supply shock lands. When μ is near zero, the inefficient risk sharing effect dominates, so cross-market impact increases in μ. When μ is near 1, the width of the pipe effect dominates, so cross-market impact declines with μ.

What is the intuition for the fact that $sign\left(\Omega_{ij}\left(\mu\right)\right)$ flips if $sign\left(\gamma_{ij}\right) \neq sign(\gamma_{ij}^P)$? In integrated markets ($\mu = 0$), the signs of cross-market price impact are determined by the signs of cross-market return correlations. In partially integrated markets ($\mu \in (0,1)$), partial-correlations—or multivariate regression coefficients—also play a role. For instance, suppose that the returns on asset 1 is positively correlated with those on assets 2 and 3. However, suppose that if we run a multivariate regression of 1 on 2 and 3, the coefficient on 2 is positive and that on 3 is negative.

Suppose that there are initially very few generalists—markets are close to completely segmented—and that there is a shock to the supply of asset 1. In highly segmented markets, this is going to have a large positive impact on the expected returns on 1 and smaller impacts on those of 2 and 3. Following the shock to the supply of asset 1, generalists want to go long 1 and to hedge this additional risk they want to short 2 and go *long* 3. To induce specialists to take the other side in markets 2 and 3, the return on 2 must rise and those on 3 must *fall*.

As we add more and more generalists who care about the comovement between the returns on the three assets, the shock to the supply of 1 pushes up the returns on both 2 and 3. Thus, as the number of generalists rises, the sign of the price impact on 3 flips signs. In the limit where everyone is a generalist, the rise in returns guarantees that generalists simply absorb the shock to the supply of 1 without trading 2 or 3.

C.2 Adding unhedged bond investors

In our Section 5.2, we add bond investors who cannot hedge foreign exchange risk—i.e., investors who cannot separately manage the FX exposure resulting from investments they make in non-local, long-term bonds. We assume there are three investor types—all with mean-variance preferences over one-period-ahead wealth and risk tolerance τ in domestic currency terms—who only differ in terms of the assets they can trade:

1. *Unhedged domestic investors* are present in mass $\eta/2$. They can trade short-term domestic bonds, long-term domestic bonds, and long-term foreign bonds, but not short-term foreign bonds. Thus, if they buy long-term foreign bonds, they must take on foreign exchange exposure, generating an excess return of $rx^{y^*}_{t+1} + rx^q_{t+1}$ over short-term domestic bonds.

2. *Unhedged foreign investors* are present in mass $\eta/2$ and are the mirror image of unhedged domestic investors. If they buy long-term domestic bonds, they must take on FX exposure, generating an excess return of $rx^y_{t+1} - rx^q_{t+1}$ over short-term foreign bonds.

3. *Global bond investors*, present in mass $(1-\eta)$, can hold short- and long-term bonds in both currencies and can engage in all three carry trades.

Unhedged investors will exhibit home bias in equilibrium. For instance, since an FX-unhedged position in long-term domestic bonds is always riskier than the FX-hedged position, it is particularly risky for foreign unhedged investors to invest in domestic bonds. Thus, relative to global rates investors and domestic unhedged investors, foreign unhedged investors face a comparative disadvantage in holding long-term domestic bonds.

C.2.1 Details and solution

Unhedged domestic investors Unhedged domestic investors are present in mass $\eta/2$. They can trade short-term domestic bonds, long-term domestic bonds, and long-term foreign bonds, but not short-term foreign bonds. Thus, if they buy long-term foreign bonds, they must take on foreign exchange exposure, generating an excess return of $rx^{y^*}_{t+1} + rx^q_{t+1}$ over short-term domestic bonds. Specifically, if unhedged domestic investors have a demand $b^{y*}_{D,t}$ for foreign long-term bonds, then their holdings of the foreign yield curve trade and the FX trade are $d^{y*}_{D,t} = b^{y*}_{D,t}$ and $d^q_{D,t} = b^{y*}_{D,t}$. So we have

$$\mathbf{d}_{D,t} = \begin{bmatrix} d^y_{D,t} \\ d^{y*}_{D,t} \\ d^q_{D,t} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b^y_{D,t} \\ b^{y*}_{D,t} \end{bmatrix} = \mathbf{H}_D \mathbf{b}_{D,t}$$

Unhedged domestic investors choose their demands $\mathbf{b}_{D,t}$ over returns

$$\mathbf{rx}_{D,t+1} = \begin{bmatrix} rx^y_{t+1} \\ rx^{y*}_{t+1} + rx^q_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} rx^y_{t+1} \\ rx^{y^*}_{t+1} \\ rx^q_{t+1} \end{bmatrix} = \mathbf{H}'_D \mathbf{rx}_{t+1}$$

Solving

$$\max_{\mathbf{b}_{D,t}} \left\{ \mathbf{b}'_{D,t} E_t \left[\mathbf{rx}_{D,t+1} \right] - \frac{\tau}{2} \mathbf{b}'_{D,t} Var_t \left[\mathbf{rx}_{D,t+1} \right] \mathbf{b}_{D,t} \right\}.$$

Thus, we have

$$\begin{aligned} \mathbf{b}_{D,t} &= \tau \left(\mathbf{H}'_D Var_t \left[\mathbf{rx}_{t+1} \right] \mathbf{H}_D \right)^{-1} \mathbf{H}'_D E_t \left[\mathbf{rx}_{t+1} \right] \\ &= \tau \frac{\begin{bmatrix} (V_q - 2C_q + V_y) \cdot E_t \left[rx^y_{t+1} \right] - (C_q + C_y) \cdot E_t \left[rx^{y*}_{t+1} + rx^q_{t+1} \right] \\ -(C_q + C_y) \cdot E_t \left[rx^y_{t+1} \right] + V_y \cdot E_t \left[rx^{y*}_{t+1} + rx^q_{t+1} \right] \end{bmatrix}}{V_y \left(V_q - 2C_q + V_y \right) - \left(C_q + C_y \right)^2} \end{aligned}$$

This implies that

$$\begin{aligned}\mathbf{d}_{D,t} &= \tau \mathbf{H}_D \left(\mathbf{H}_D' Var_t\left[\mathbf{rx}_{t+1}\right]\mathbf{H}_D\right)^{-1}\mathbf{H}_D' E_t\left[\mathbf{rx}_{t+1}\right] \\ &= \tau \frac{\begin{bmatrix} (V_q - 2C_q + V_y)\cdot E_t\left[rx_{t+1}^{y}\right] - (C_q + C_y)\cdot E_t\left[rx_{t+1}^{y*} + rx_{t+1}^{q}\right] \\ -(C_q + C_y)\cdot E_t\left[rx_{t+1}^{y}\right] + V_y \cdot E_t\left[rx_{t+1}^{y*} + rx_{t+1}^{q}\right] \\ -(C_q + C_y)\cdot E_t\left[rx_{t+1}^{y}\right] + V_y \cdot E_t\left[rx_{t+1}^{y*} + rx_{t+1}^{q}\right]\end{bmatrix}}{V_y\left(V_q - 2C_q + V_y\right) - \left(C_q + C_y\right)^2}\end{aligned}$$

Unhedged foreign investors Unhedged foreign investors are present in mass $\eta/2$ and are the mirror image of unhedged domestic investors. They can buy long-term foreign bonds, long-term domestics bonds, but not short-term domestic bonds. Thus, if they buy long-term domestic bonds, they must take on FX exposure, generating an excess return of $rx_{t+1}^{y} - rx_{t+1}^{q}$ over short-term foreign bonds.

Specifically, if unhedged foreign investors have a demand $b_{F,t}^{y}$ for domestic long-term bonds, they we will have $d_{F,t}^{y} = b_{D,t}^{y}$ and $d_{F,t}^{q} = -b_{F,t}^{y}$. So we have

$$\mathbf{d}_{F,t} = \begin{bmatrix} d_{F,t}^{y} \\ d_{F,t}^{y*} \\ d_{F,t}^{q}\end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0\end{bmatrix}\begin{bmatrix} b_{F,t}^{y} \\ b_{F,t}^{y*}\end{bmatrix} = \mathbf{H}_F \mathbf{b}_{F,t}$$

Think of unhedged foreign investors as picking demands $\mathbf{b}_{F,t}$ over returns

$$\mathbf{rx}_{F,t+1} = \begin{bmatrix} rx_{t+1}^{y} - rx_{t+1}^{q} \\ rx_{t+1}^{y*}\end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix} rx_{t+1}^{y} \\ rx_{t+1}^{y*} \\ rx_{t+1}^{q}\end{bmatrix} = \mathbf{H}_F' \mathbf{rx}_{t+1}$$

Thus, unhedged foreign investors solve

$$\max_{\mathbf{b}_{F,t}}\left\{\mathbf{b}_{F,t}' E_t\left[\mathbf{rx}_{F,t+1}\right] - \frac{\tau}{2}\mathbf{b}_{F,t}' Var_t\left[\mathbf{rx}_{F,t+1}\right]\mathbf{b}_{F,t}\right\}.$$

Thus, we have

$$\begin{aligned}\mathbf{b}_{F,t} &= \tau\left(\mathbf{H}_F' Var_t\left[\mathbf{rx}_{t+1}\right]\mathbf{H}_F\right)^{-1}\mathbf{H}_F' E_t\left[\mathbf{rx}_{t+1}\right] \\ &= \tau\frac{\begin{bmatrix} V_y \cdot E_t\left[rx_{t+1}^{y} - rx_{t+1}^{q}\right] - (C_q + C_y)\cdot E_t\left[rx_{t+1}^{y*}\right] \\ (V_q - 2C_q + V_y)\cdot E_t\left[rx_{t+1}^{y*}\right] - (C_q + C_y)\cdot E_t\left[rx_{t+1}^{y} - rx_{t+1}^{q}\right]\end{bmatrix}}{V_y\left(V_q - 2C_q + V_y\right) - \left(C_q + C_y\right)^2}\end{aligned}$$

and

$$\begin{aligned}\mathbf{d}_{F,t} &= \tau \mathbf{H}_F\left(\mathbf{H}_F' Var_t\left[\mathbf{rx}_{t+1}\right]\mathbf{H}_F\right)^{-1}\mathbf{H}_F' E_t\left[\mathbf{rx}_{t+1}\right] \\ &= \tau\frac{\begin{bmatrix} V_y \cdot E_t\left[rx_{t+1}^{y} - rx_{t+1}^{q}\right] - (C_q + C_y)\cdot E_t\left[rx_{t+1}^{y*}\right] \\ (V_q - 2C_q + V_y)\cdot E_t\left[rx_{t+1}^{y*}\right] - (C_q + C_y)\cdot E_t\left[rx_{t+1}^{y} - rx_{t+1}^{q}\right] \\ (C_q + C_y)\cdot E_t\left[rx_{t+1}^{y*}\right] - V_y \cdot E_t\left[rx_{t+1}^{y} - rx_{t+1}^{q}\right]\end{bmatrix}}{V_y\left(V_q - 2C_q + V_y\right) - \left(C_q + C_y\right)^2}\end{aligned}$$

Global bond investors Global bond investors, present in mass $(1-\eta)$, can hold short- and long-term bonds in both currencies and can engage in all three carry trades. Thus, global investors have demands

$$\mathbf{d}_{G,t} = \tau\left(Var_t\left[\mathbf{rx}_{t+1}\right]\right)^{-1}E_t\left[\mathbf{rx}_{t+1}\right]$$

Market clearing and solution Letting $\mathbf{V} = Var_t[\mathbf{rx}_{t+1}]$, the market clearing condition

$$\begin{aligned}\mathbf{s}_t &= \frac{\eta}{2}\mathbf{d}_{D,t} + \frac{\eta}{2}\mathbf{d}_{F,t} + (1-\eta)\,\mathbf{d}_{G,t} \\ &= \frac{\eta}{2}\tau\mathbf{H}_D\left(\mathbf{H}_D'\mathbf{V}\mathbf{H}_D\right)^{-1}\mathbf{H}_D' E_t[\mathbf{rx}_{t+1}] + \frac{\eta}{2}\tau\mathbf{H}_F\left(\mathbf{H}_F'\mathbf{V}\mathbf{H}_F\right)^{-1}\mathbf{H}_F' E_t[\mathbf{rx}_{t+1}] + (1-\eta)\,\tau\mathbf{V}^{-1}E_t[\mathbf{rx}_{t+1}],\end{aligned}$$

so we have

$$E_t[\mathbf{rx}_{t+1}] = \tau^{-1}\left(\frac{\eta}{2}\mathbf{H}_D\left(\mathbf{H}_D'\mathbf{V}(\eta)\mathbf{H}_D\right)^{-1}\mathbf{H}_D' + \frac{\eta}{2}\mathbf{H}_F\left(\mathbf{H}_F'\mathbf{V}(\eta)\mathbf{H}_F\right)^{-1}\mathbf{H}_F' + (1-\eta)\left[\mathbf{V}(\eta)\right]^{-1}\right)^{-1}\mathbf{s}_t.$$

Fixed point problem The fixed point for $\mathbf{A}$ is

$$\mathbf{A} = \left[\tau^{-1}\left(\frac{\eta}{2}\cdot\mathbf{H}_D\left(\mathbf{H}_D'\mathbf{V}\mathbf{H}_D\right)^{-1}\mathbf{H}_D' + \frac{\eta}{2}\cdot\mathbf{H}_F\left(\mathbf{H}_F'\mathbf{V}\mathbf{H}_F\right)^{-1}\mathbf{H}_F' + (1-\eta)\,\mathbf{V}^{-1}\right)^{-1}\mathbf{S}_1 - \mathbf{R}_1\right]\oslash[\mathbf{B}_0\mathbf{E} + \mathbf{B}_1\mathbf{E}\boldsymbol{\Phi}]$$

where $\oslash$ denotes element-wise matrix division (i.e., Hadamard division) and

$$\mathbf{V} = \mathbf{B}_1\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}'\mathbf{B}_1'.$$

We can think of equilibrium as a fixed point problem involving the return impact matrix, $\boldsymbol{\Theta}$, that maps changes in asset to supply to shifts in asset returns—i.e. $E_t[\mathbf{rx}_{t+1}] = \boldsymbol{\Theta}\mathbf{s}_t$. This return impact matrix is given by

$$\boldsymbol{\Theta} = \tau^{-1}\left(\frac{\eta}{2}\mathbf{H}_D\left(\mathbf{H}_D'\mathbf{V}\mathbf{H}_D\right)^{-1}\mathbf{H}_D' + \frac{\eta}{2}\mathbf{H}_F\left(\mathbf{H}_F'\mathbf{V}\mathbf{H}_F\right)^{-1}\mathbf{H}_F' + (1-\eta)\left[\mathbf{V}\right]^{-1}\right)^{-1}$$

Since $\mathbf{A}_s = \boldsymbol{\Theta}\oslash\mathbf{Z}_s$ where $\mathbf{Z}_s = [\mathbf{B}_0\mathbf{E} + \mathbf{B}_1\mathbf{E}\boldsymbol{\Phi}]^{[3-5]}$, we have

$$\mathbf{V} = \mathbf{V}_i + (\mathbf{B}_1\boldsymbol{\Theta}\oslash\mathbf{Z}_s)\,\boldsymbol{\Sigma}_s\,(\mathbf{B}_1\boldsymbol{\Theta}\oslash\mathbf{Z}_s)' = \mathbf{V}_i + \mathbf{Z}\circ\boldsymbol{\Theta}\boldsymbol{\Sigma}_s\boldsymbol{\Theta}\circ\mathbf{Z}',$$

where $\mathbf{Z}$ satisfies $(\mathbf{B}_1\boldsymbol{\Theta}\oslash\mathbf{Z}_s) = \boldsymbol{\Theta}\circ\mathbf{Z}$. Even though $\boldsymbol{\Theta}$ is symmetric, $\mathbf{Z}$ is not, so $\boldsymbol{\Theta}\circ\mathbf{Z}$ is not symmetric. Thus, the relevant fixed point problem in $\boldsymbol{\Theta}$ is

$$\boldsymbol{\Theta} = \tau^{-1}\begin{pmatrix} \frac{\eta}{2}\mathbf{H}_D\left(\mathbf{H}_D'[\mathbf{V}_i + \mathbf{Z}\circ\boldsymbol{\Theta}\boldsymbol{\Sigma}_s\boldsymbol{\Theta}\circ\mathbf{Z}']\mathbf{H}_D\right)^{-1}\mathbf{H}_D' \\ +\frac{\eta}{2}\mathbf{H}_F\left(\mathbf{H}_F'[\mathbf{V}_i + \mathbf{Z}\circ\boldsymbol{\Theta}\boldsymbol{\Sigma}_s\boldsymbol{\Theta}\circ\mathbf{Z}']\mathbf{H}_F\right)^{-1}\mathbf{H}_F' \\ +(1-\eta)\left[\mathbf{V}_i + \mathbf{Z}\circ\boldsymbol{\Theta}\boldsymbol{\Sigma}_s\boldsymbol{\Theta}\circ\mathbf{Z}'\right]^{-1} \end{pmatrix}^{-1}.$$

As always, we focus on the unique stable solution to this fixed point problem.

Trading behavior Writing out $\boldsymbol{\Theta}$, we have

$$\boldsymbol{\Theta} = \tau^{-1}\left(\frac{\eta}{2}\mathbf{H}_D\left(\mathbf{H}_D'\mathbf{V}\mathbf{H}_D\right)^{-1}\mathbf{H}_D' + \frac{\eta}{2}\mathbf{H}_F\left(\mathbf{H}_F'\mathbf{V}\mathbf{H}_F\right)^{-1}\mathbf{H}_F' + (1-\eta)\,\mathbf{V}^{-1}\right)^{-1}.$$

Thus, we have

$$\begin{aligned}
\mathbf{I} &\neq \frac{\partial \mathbf{d}_{D,t}}{\partial \mathbf{s}_t} = \tau \mathbf{H}_D \left(\mathbf{H}_D' \mathbf{V}(\eta \mathbf{H}_D) \right)^{-1} \mathbf{H}_D' \mathbf{\Theta}, \\
\mathbf{I} &\neq \frac{\partial \mathbf{d}_{F,t}}{\partial \mathbf{s}_t} = \tau \mathbf{H}_F \left(\mathbf{H}_F' \mathbf{V} \mathbf{H}_F \right)^{-1} \mathbf{H}_F' \mathbf{\Theta}, \\
\mathbf{I} &\neq \frac{\partial \mathbf{d}_{G,t}}{\partial \mathbf{s}_t} = \tau \mathbf{V}^{-1} \mathbf{\Theta},
\end{aligned}$$

so the first part of the Proposition follows trivially.

Understanding how $\mathbf{\Theta}(\eta)$ varies as a function of η. We first want to understand how this solution $\mathbf{\Theta}(\eta)$ varies as a function of η. Clearly, $\mathbf{\Theta}(\eta)$ is positive definite. However, we are interested in understanding when/whether $\mathbf{\Theta}'(\eta) = \partial \mathbf{\Theta}(\eta) / \partial \eta$ is itself a positive definite matrix.

We have

$$\mathbf{\Theta}(\eta) = \tau^{-1} \left(\frac{\eta}{2} \mathbf{H}_D \left(\mathbf{H}_D' \mathbf{V}(\eta) \mathbf{H}_D \right)^{-1} \mathbf{H}_D' + \frac{\eta}{2} \mathbf{H}_F \left(\mathbf{H}_F' \mathbf{V}(\eta) \mathbf{H}_F \right)^{-1} \mathbf{H}_F' + (1-\eta) \left[\mathbf{V}(\eta) \right]^{-1} \right)^{-1}.$$

This implies that

$$\begin{aligned}
\mathbf{\Theta}'(\eta) &= \tau \mathbf{\Theta}(\eta) \left\{ \left[\mathbf{V}(\eta) \right]^{-1} - \frac{1}{2} \mathbf{H}_D \left(\mathbf{H}_D' \mathbf{V}(\eta) \mathbf{H}_D \right)^{-1} \mathbf{H}_D' - \frac{1}{2} \mathbf{H}_F \left(\mathbf{H}_F' \mathbf{V}(\eta) \mathbf{H}_F \right)^{-1} \mathbf{H}_F' \right\} \mathbf{\Theta}(\eta) \\
&\quad + \tau \mathbf{\Theta}(\eta) \left[\begin{array}{c} \frac{\eta}{2} \mathbf{H}_D \left(\mathbf{H}_D' \mathbf{V}(\eta) \mathbf{H}_D \right)^{-1} \left(\mathbf{H}_D' \mathbf{V}'(\eta) \mathbf{H}_D \right) \left(\mathbf{H}_D' \mathbf{V}(\eta) \mathbf{H}_D \right)^{-1} \mathbf{H}_D' \\ + \frac{\eta}{2} \mathbf{H}_F \left(\mathbf{H}_F' \mathbf{V}(\eta) \mathbf{H}_F \right)^{-1} \left(\mathbf{H}_F' \mathbf{V}'(\eta) \mathbf{H}_F \right) \left(\mathbf{H}_F' \mathbf{V}(\eta) \mathbf{H}_F \right)^{-1} \mathbf{H}_F' \\ + (1-\eta) \left[\mathbf{V}(\eta) \right]^{-1} \mathbf{V}'(\eta) \left[\mathbf{V}(\eta) \right]^{-1} \end{array} \right] \mathbf{\Theta}(\eta). \qquad (67)
\end{aligned}$$

As above, we assume that FX rates are subject to some FX-specific fundamental risk, so $\mathbf{V}^{-1}$ exists even in the absence of supply risk. In this case, we can show that $\{\mathbf{V}^{-1} - \frac{1}{2} \mathbf{H}_D \left(\mathbf{H}_D' \mathbf{V}(\eta) \mathbf{H}_D \right)^{-1} \mathbf{H}_D' - \frac{1}{2} \mathbf{H}_F \left(\mathbf{H}_F' \mathbf{V} \mathbf{H}_F \right)^{-1} \mathbf{H}_F'\}$ is positive semi-definite, immediately implying that the first matrix in curly braces in equation (67) positive semi-definite. This means that $\mathbf{\Theta}'(\eta)$ must always be positive definite in the absence of supply risk. Furthermore, by continuity of the stable equilibrium in the model's underlying parameters, $\mathbf{\Theta}'(\eta)$ must continue to be positive-definite when supply risk is small ($\mathbf{\Sigma}_s$ is small).

Proof that $\mathbf{\Theta}'(\eta)$ is positive semi-definite in the absence of supply risk. To prove that $\{\mathbf{V}^{-1} - \frac{1}{2} \mathbf{H}_D \left(\mathbf{H}_D' \mathbf{V}(\eta) \mathbf{H}_D \right)^{-1} \mathbf{H}_D' - \frac{1}{2} \mathbf{H}_F \left(\mathbf{H}_F' \mathbf{V} \mathbf{H}_F \right)^{-1} \mathbf{H}_F'\}$ is positive semi-definite it suffices to prove that $[\frac{1}{2}\mathbf{H}_D\left(\mathbf{H}_D'\mathbf{V}(\eta)\mathbf{H}_D\right)^{-1}\mathbf{H}_D' + \frac{1}{2}\mathbf{H}_F\left(\mathbf{H}_F'\mathbf{V}\mathbf{H}_F\right)^{-1}\mathbf{H}_F']^{-1} - \mathbf{V}$ is positive semi-definite.[13] We have

$$\begin{aligned}
&\frac{1}{2}\mathbf{H}_D\left(\mathbf{H}_D'\mathbf{V}\mathbf{H}_D\right)^{-1}\mathbf{H}_D' + \frac{1}{2}\mathbf{H}_F\left(\mathbf{H}_F'\mathbf{V}\mathbf{H}_F\right)^{-1}\mathbf{H}_F' \\
= \;& \frac{1}{V_y\left(V_y+V_q-2C_q\right)-\left(C_q+C_y\right)^2}\begin{bmatrix} \left(V_y-C_q\right)+\frac{1}{2}V_q & -\left(C_q+C_y\right) & -\frac{1}{2}\left(C_q+C_y+V_y\right) \\ -\left(C_q+C_y\right) & \left(V_y-C_q\right)+\frac{1}{2}V_q & \frac{1}{2}\left(C_q+C_y+V_y\right) \\ -\frac{1}{2}\left(C_q+C_y+V_y\right) & \frac{1}{2}\left(C_q+C_y+V_y\right) & V_y \end{bmatrix}
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\left[\frac{1}{2}\mathbf{H}_D\left(\mathbf{H}_D'\mathbf{V}\mathbf{H}_D\right)^{-1}\mathbf{H}_D' + \frac{1}{2}\mathbf{H}_F\left(\mathbf{H}_F'\mathbf{V}\mathbf{H}_F\right)^{-1}\mathbf{H}_F'\right]^{-1} - \mathbf{V} \\
= &\begin{bmatrix} -\frac{C_q^2+2C_qC_y+2C_qV_y+C_y^2-V_y^2-V_qV_y}{V_q-2C_y-4C_q+2V_y} & -\frac{C_q^2-2C_qC_y-2C_qV_y-C_y^2+V_qC_y+V_y^2}{V_q-2C_y-4C_q+2V_y} & \left(C_y+V_y\right) \\ -\frac{C_q^2-2C_qC_y-2C_qV_y-C_y^2+V_qC_y+V_y^2}{V_q-2C_y-4C_q+2V_y} & -\frac{C_q^2+2C_qC_y+2C_qV_y+C_y^2-V_y^2-V_qV_y}{V_q-2C_y-4C_q+2V_y} & -\left(C_y+V_y\right) \\ \left(C_y+V_y\right) & -\left(C_y+V_y\right) & 2\left(C_y+V_y\right) \end{bmatrix}
\end{aligned}$$

The eigenvalues of this later matrix are:

1. $\lambda_1 = 3\left(V_y+C_y\right) > 0$. The inequality follows from the fact that $\left(V_y+C_y\right) > 0$..
2. $\lambda_2 = [V_q\left(V_y-C_y\right)-2C_q^2]/[2\left(V_y-C_y\right)+V_q-4C_q] > 0$.
   - Since $\mathbf{V}$ is positive definite, we have $\det(\mathbf{V}) = \left(C_y+V_y\right)\left(V_q\left(V_y-C_y\right)-2C_q^2\right) > 0$. Then, since $\left(C_y+V_y\right) > 0$, we have $V_q\left(V_y-C_y\right)-2C_q^2 > 0$.
   - Since $\mathbf{V}$ is positive definite, we have $2\left(V_y-C_y\right)+V_q-4C_q = Var\left[\left(rx_{t+1}^{y*}-rx_{t+1}^{y}\right)+rx_{t+1}^{q}\right] > 0$.
3. $\lambda_3 = 0$.

Thus, we conclude that $\{\mathbf{V}^{-1} - \frac{1}{2}\mathbf{H}_D\left(\mathbf{H}_D'\mathbf{V}(\eta)\mathbf{H}_D\right)^{-1}\mathbf{H}_D' - \frac{1}{2}\mathbf{H}_F\left(\mathbf{H}_F'\mathbf{V}\mathbf{H}_F\right)^{-1}\mathbf{H}_F'\}$ is positive semi-definite.

How individual elements of $\mathbf{\Theta}(\eta)$ behave as a function of η. We consider the special case without supply risk. Here we have

$$\mathbf{V} = \begin{bmatrix} \left(\frac{\delta}{1-\delta\phi_i}\right)^2\sigma_i^2 & \left(\frac{\delta}{1-\delta\phi_i}\right)^2\rho\sigma_i^2 & \frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2\left(1-\rho\right) \\ \left(\frac{\delta}{1-\delta\phi_i}\right)^2\rho\sigma_i^2 & \left(\frac{\delta}{1-\delta\phi_i}\right)^2\sigma_i^2 & -\frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2\left(1-\rho\right) \\ \frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2\left(1-\rho\right) & -\frac{\delta}{1-\delta\phi_i}\frac{1}{1-\phi_i}\sigma_i^2\left(1-\rho\right) & \sigma_{q\infty}^2+\left(\frac{1}{1-\phi_i}\right)^2 2\sigma_i^2\left(1-\rho\right) \end{bmatrix},$$

which is independent of η. Assuming $\sigma_{q\infty}^2 > 0$, $\mathbf{V}$ is positive definite and we have $\det(\mathbf{V}) > 0$. We have

$$\mathbf{\Theta}(\eta) = \tau^{-1}\left(\frac{\eta}{2}\mathbf{H}_D\left(\mathbf{H}_D'\mathbf{V}\mathbf{H}_D\right)^{-1}\mathbf{H}_D' + \frac{\eta}{2}\mathbf{H}_F\left(\mathbf{H}_F'\mathbf{V}\mathbf{H}_F\right)^{-1}\mathbf{H}_F' + (1-\eta)\left[\mathbf{V}\right]^{-1}\right)^{-1}.$$

[13] *Suppose that $\mathbf{A}$ and $\mathbf{B}$ are positive definite matrices. If $\mathbf{A} \prec \mathbf{B}$, then $\mathbf{B}^{-1} \prec \mathbf{A}^{-1}$. (We use $\mathbf{A} \prec \mathbf{B}$ to mean that $(\mathbf{B}-\mathbf{A})$ is positive-definite.) Similarly, if $\mathbf{A} \preceq \mathbf{B}$, then $\mathbf{B}^{-1} \preceq \mathbf{A}^{-1}$. (We use $\mathbf{A} \preceq \mathbf{B}$ to mean that $(\mathbf{B}-\mathbf{A})$ is positive semi-definite.)* To prove this claim, first note that $\mathbf{A} \prec \mathbf{B} \Rightarrow \mathbf{CAC}' \prec \mathbf{CBC}'$ for any conformable matrix $\mathbf{C}$. Second, note that $\mathbf{I} \prec \mathbf{B} \Rightarrow \mathbf{B}^{-1} \prec \mathbf{I}$. To see this, note that $\mathbf{B}^{-1} = \mathbf{B}^{-1/2}\mathbf{I}\mathbf{B}^{-1/2} \prec \mathbf{B}^{-1/2}\mathbf{B}\mathbf{B}^{-1/2} = \mathbf{I}$. Third, note that $\mathbf{A} \prec \mathbf{B} \Rightarrow \mathbf{0} \prec \mathbf{B}-\mathbf{A} \Rightarrow \mathbf{0} \prec \mathbf{A}^{-1/2}\left(\mathbf{B}-\mathbf{A}\right)\mathbf{A}^{-1/2} \Rightarrow \mathbf{I} \prec \mathbf{A}^{-1/2}\mathbf{B}\mathbf{A}^{-1/2}$. Thus, we have $\mathbf{A}^{1/2}\mathbf{B}^{-1}\mathbf{A}^{1/2} = \left(\mathbf{A}^{-1/2}\mathbf{B}\mathbf{A}^{-1/2}\right)^{-1} \prec \mathbf{I}$. Finally, we have $\mathbf{B}^{-1} = \mathbf{A}^{-1/2}(\mathbf{A}^{1/2}\mathbf{B}^{-1}\mathbf{A}^{1/2})\mathbf{A}^{-1/2} \prec \mathbf{A}^{-1/2}(\mathbf{I})\mathbf{A}^{-1/2} = \mathbf{A}^{-1}$.

$\mathbf{\Theta}(\eta)$ is positive definite and $\mathbf{\Theta}'(\eta)$ is positive semi-definite.

Diagonal elements of $\mathbf{\Theta}'(\eta)$ Since $\mathbf{\Theta}'(\eta)$ is positive semi-definite, it follows that $\mathbf{\Theta}'_{[1,1]}(\eta) \geq 0$, $\mathbf{\Theta}'_{[2,2]}(\eta) \geq 0$, $\mathbf{\Theta}'_{[3,3]}(\eta) \geq 0$ for all η.

Computing $\mathbf{\Theta}(1) - \mathbf{\Theta}(0) = \int_0^1 \mathbf{\Theta}'(\eta)\, d\eta$. We have

$$\mathbf{\Theta}(0) = \tau^{-1} \begin{bmatrix} V_y & C_y & C_q \\ C_y & V_y & -C_q \\ C_q & -C_q & V_q \end{bmatrix},$$

and

$$\mathbf{\Theta}(1) = \tau^{-1} \begin{bmatrix} \frac{3V_y^2+2V_qV_y-C_q^2-2C_qC_y-6C_qV_y-C_y^2-2C_yV_y}{2(V_y-C_y)+V_q-4C_q} & -\frac{(C_q+C_y-V_y)^2}{2(V_y-C_y)+V_q-4C_q} & C_q+(V_y+C_y) \\ -\frac{(C_q+C_y-V_y)^2}{2(V_y-C_y)+V_q-4C_q} & \frac{3V_y^2+2V_qV_y-C_q^2-2C_qC_y-6C_qV_y-C_y^2-2C_yV_y}{2(V_y-C_y)+V_q-4C_q} & -C_q-(V_y+C_y) \\ C_q+(V_y+C_y) & -C_q-(V_y+C_y) & V_q+2(V_y+C_y) \end{bmatrix}$$

Thus, we have

$$\mathbf{\Theta}(1) - \mathbf{\Theta}(0) = \tau^{-1} \begin{bmatrix} \frac{V_y(V_y+V_q-2C_q)-(C_q+C_y)^2}{2(V_y-C_y)+V_q-4C_q} & -\frac{(C_q+C_y-V_y)^2}{2V_y+V_q-2C_y-4C_q} - C_y & C_y+V_y \\ -\frac{(C_q+C_y-V_y)^2}{2V_y+V_q-2C_y-4C_q} - C_y & \frac{V_y(V_y+V_q-2C_q)-(C_q+C_y)^2}{2(V_y-C_y)+V_q-4C_q} & -(C_y+V_y) \\ C_y+V_y & -(C_y+V_y) & 2(V_y+C_y) \end{bmatrix} \propto \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix},$$

which follows from the facts that

$$\begin{aligned} \frac{V_y(V_y+V_q-2C_q)-(C_q+C_y)^2}{2(V_y-C_y)+V_q-4C_q} &= \frac{\sigma_{q\infty}^2 \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \sigma_i^2 + \delta^2\sigma_i^4 \frac{(1-\delta)^2}{(1-\phi_i)^2}\frac{1-\rho^2}{(1-\delta\phi_i)^4}}{\sigma_{q\infty}^2 + 2\sigma_i^2(1-\rho)\left(\frac{\delta}{1-\delta\phi_i} - \frac{1}{1-\phi_i}\right)^2} > 0, \\ -\frac{(C_q+C_y-V_y)^2}{2V_y+V_q-2C_y-4C_q} - C_y &= -\frac{\left(\frac{\sigma_i^2\delta(1-\delta)(1-\rho)}{(1-\phi_i)(1-\delta\phi_i)^2}\right)^2}{\sigma_{q\infty}^2 + 2\sigma_i^2(1-\rho)\left(\frac{\delta}{1-\delta\phi_i} - \frac{1}{1-\phi_i}\right)^2} - \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \rho\sigma_i^2 < 0, \\ V_y + C_y &= \left(\frac{\delta}{1-\delta\phi_i}\right)^2 \sigma_i^2(1+\rho) > 0. \end{aligned}$$

Computing $\mathbf{\Theta}'(0)$. Compute derivative at $\eta = 0$ where $\mathbf{\Theta}(\eta) = \tau^{-1}\mathbf{V}$. We have

$$\begin{aligned} \mathbf{\Theta}'(0) &= \tau^{-1}\mathbf{V}\left[\mathbf{V}^{-1} - \frac{1}{2}\mathbf{H}_D(\mathbf{H}_D'\mathbf{V}\mathbf{H}_D)^{-1}\mathbf{H}_D' - \frac{1}{2}\mathbf{H}_F(\mathbf{H}_F'\mathbf{V}\mathbf{H}_F)^{-1}\mathbf{H}_F'\right]\mathbf{V} \\ &= \tau^{-1}\frac{(C_y+V_y)\left(V_q(V_y-C_y)-2C_q^2\right)}{(C_q+C_y)^2+V_y(V_y+V_q-2C_q)} \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} \propto \begin{bmatrix} + & 0 & + \\ 0 & + & - \\ + & - & + \end{bmatrix}, \end{aligned}$$

which follows from that facts that $(C_y+V_y)\left(V_q(V_y-C_y)-2C_q^2\right) = \det(V) > 0$ and $V_y+V_q-2C_q = Var\left[rx_{t+1}^y - rx_{t+1}^q\right] > 0$.

Computing $\boldsymbol{\Theta}'(1)$. Compute the derivative at $\eta = 1$. We have

$$\boldsymbol{\Theta}(1) = \tau^{-1} \begin{bmatrix} \frac{3V_y^2+2V_qV_y-C_q^2-2C_qC_y-6C_qV_y-C_y^2-2C_yV_y}{2(V_y-C_y)+V_q-4C_q} & -\frac{(C_q+C_y-V_y)^2}{2(V_y-C_y)+V_q-4C_q} & C_q+(V_y+C_y) \\ -\frac{(C_q+C_y-V_y)^2}{2(V_y-C_y)+V_q-4C_q} & \frac{3V_y^2+2V_qV_y-C_q^2-2C_qC_y-6C_qV_y-C_y^2-2C_yV_y}{2(V_y-C_y)+V_q-4C_q} & -C_q-(V_y+C_y) \\ C_q+(V_y+C_y) & -C_q-(V_y+C_y) & V_q+2(V_y+C_y) \end{bmatrix}.$$

We have

$$\begin{aligned} \boldsymbol{\Theta}'_{[3,1]}(1) &= \tau^{-1} 2 \frac{(C_y+V_y)\left(V_y(V_y+V_q-2C_q)-(C_q+C_y)^2\right)}{V_q(V_y-C_y)-2C_q^2} \\ &= \tau^{-1} 2 \frac{(C_y+V_y)\left(Var\left[rx_{t+1}^y\right]Var\left[rx_{t+1}^{y*}+rx_{t+1}^q\right]-\left(Cov\left[rx_{t+1}^y, rx_{t+1}^{y*}+rx_{t+1}^q\right]\right)^2\right)}{V_q(V_y-C_y)-2C_q^2} > 0. \end{aligned}$$

Of course, we have $\boldsymbol{\Theta}'_{[3,1]}(1) = \boldsymbol{\Theta}'_{[1,3]}(1) = -\boldsymbol{\Theta}'_{[3,2]}(1) = -\boldsymbol{\Theta}'_{[2,3]}(1)$. We also have

$$\boldsymbol{\Theta}'_{[2,1]}(1) = -4\tau^{-1}\left(\frac{C_q^2+2C_qC_y+2C_qV_y+C_y^2-V_y^2-V_qV_y}{V_q-2C_y-4C_q+2V_y}\right)^2 \frac{\left(C_q^2-2C_qC_y-2C_qV_y-C_y^2+V_qC_y+V_y^2\right)}{(C_y+V_y)\left(V_q(V_y-C_y)-2C_q^2\right)}.$$

Thus, $\boldsymbol{\Theta}'_{[2,1]}(1)$ has the opposite sign of

$$C_q^2-2C_qC_y-2C_qV_y-C_y^2+V_qC_y+V_y^2 = \sigma_{q\infty}^2\left(\frac{\delta}{1-\delta\phi_i}\right)^2\rho\sigma_i^2+\delta^2\sigma_i^4\frac{(1-\delta)^2}{(1-\phi_i)^2}\frac{1-\rho^2}{(1-\delta\phi_i)^4} > 0.$$

Thus, we have $\boldsymbol{\Theta}'_{[2,1]}(1) < 0$. In summary, we conclude that

$$\boldsymbol{\Theta}'(1) \propto \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}.$$

$\boldsymbol{\Theta}'(\eta)$ for $\eta \in [0,1]$ Although the algebra gets extremely messy, we can show that

$$\boldsymbol{\Theta}'(\eta) \propto \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

for $\eta \in [0,1]$. For simplicity, we show this explicitly below the case where $\delta \to 1$.

Computing $\lim_{\delta\to 1}\boldsymbol{\Theta}'(\eta)$ We now compute the result in the limit where $\delta \to 1$. This limit—the limit where the duration of long-term bonds becomes infinite—simplifies the algebra considerably, but does not change the underlying economics. We have $\lim_{\delta\to 1}\boldsymbol{\Theta}'(\eta) = \partial\left[\lim_{\delta\to 1}\boldsymbol{\Theta}(\eta)\right]/\partial\eta$. We now compute $\lim_{\delta\to 1}\boldsymbol{\Theta}(\eta)$.

$$\lim_{\delta\to 1}\boldsymbol{\Theta}(\eta) = \tau^{-1}\begin{bmatrix} 2\frac{\sigma_i^2}{(1-\phi_i)^2}\frac{2-\eta-\eta\rho^2}{4-4\eta+\eta^2(1-\rho^2)} & \frac{4\rho\sigma_i^2}{(1-\phi_i)^2}\frac{1-\eta}{4-4\eta+\eta^2(1-\rho^2)} & \frac{2\sigma_i^2(1-\rho)}{(1-\phi_i)^2}\frac{1}{2-\eta-\eta\rho} \\ \frac{4\rho\sigma_i^2}{(1-\phi_i)^2}\frac{1-\eta}{4-4\eta+\eta^2(1-\rho^2)} & 2\frac{\sigma_i^2}{(1-\phi_i)^2}\frac{2-\eta-\eta\rho^2}{4-4\eta+\eta^2(1-\rho^2)} & -\frac{2\sigma_i^2(1-\rho)}{(1-\phi_i)^2}\frac{1}{2-\eta-\eta\rho} \\ \frac{2\sigma_i^2(1-\rho)}{(1-\phi_i)^2}\frac{1}{2-\eta-\eta\rho} & -\frac{2\sigma_i^2(1-\rho)}{(1-\phi_i)^2}\frac{1}{2-\eta-\eta\rho} & \frac{\left(4\sigma_i^2(1-\rho)+2\sigma_{q\infty}^2(1-\phi_i)^2\right)-\eta\sigma_{q\infty}^2(1+\rho)(1-\phi_i)^2}{(1-\phi_i)^2(2-\eta(1+\rho))} \end{bmatrix}$$

Differentiating the above result, we see that

$$\lim_{\delta \to 1} \boldsymbol{\Theta}'(\eta) = \frac{\partial}{\partial \eta} \lim_{\delta \to 1} \boldsymbol{\Theta}(\eta) \propto \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}.$$